L^∞ error estimates of discontinuous Galerkin methods for the Poisson equation on a polygonal domain

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SIAM: East Asian Section Conference 2017
at Seoul National University, Seoul, South Korea
Jun. 23, 2017

1 . Introduction

Introduction

Discontinuous Galerkin (DG) method

- DG method is proposed by Reed and Hill.
- The solution is approximated by discontinuous piecewise polynomials, and its continuity between each element is controlled by numerical flux.
- L^2 norm and energy norm: many results ([ABCM02], [AM09], etc.)
- L^p norm: [CC04] (L^∞norm, smooth domain), fewer results

$\rightarrow$ There are a few applications to nonlinear problem using DG method.

In FEM, there exist many works using L^p norm.

Using method [Sch80], we analyze L^∞ norm on a polygonal domain without assuming the convexity of the shape of domain.

2 . Preliminaries

Poisson equation

Poisson equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

$\Omega \subset \mathbb{R}^2$: polygonal domain without assumption convexity
$f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$

For $g = 0$, its weak solution $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla u \cdot \nabla v dx = (f, v)_\Omega \quad {}^\forall v \in H_0^1(\Omega). \tag{2}$$

Poisson equation

Let α be a maximum angle of Ω, and $\beta = \pi/\alpha$.

Prop. 1 ([Gri76])

If $u \in H_0^1(\Omega)$ satisfies (2), then following properties hold.

(i) If Ω is a convex polygonal domain ($\beta > 1$), then $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and u satisfies

$$|u|_{H^2(\Omega)} \leq \|f\|_{L^2(\Omega)} .$$

(ii) If Ω is an non-convex polygonal domain ($1/2 < \beta < 1$) and $f \in L^p(\Omega)$ for some $1 < p < 2/(2-\beta)$, then $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ and there exists a positive constant C depending only on Ω and p such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)} .$$

Notations

$\mathcal{T}_h$:shape-regular and quasi-uniform triangulation of Ω, i.e. there exists C such that

$$\frac{h_K}{\rho_K} \leq C\,,\ \frac{h}{h_K} \leq C. \tag{3}$$

($h_K := \operatorname{diam} K$, ρ_K: radius of inscribed circle of K, $h := \max_{K\in\mathcal{T}_h} h_K$)
For $x_0 \in \Omega$ and $d > 0$, $S_d(x_0) := \{x \in \Omega \colon\ |x - x_0| < d\}$,
$d(\Omega_0, \Omega_1) := \operatorname{dist}(\Omega_0, \partial\Omega_1)$, $d_\Omega(\Omega_0, \Omega_1) := \operatorname{dist}(\Omega_0, \partial\Omega_1 \setminus \partial\Omega)$.

Function Spaces

$V^p := \{v \in L^p(\Omega) \colon v|_K \in W^{1,p}(K), (\nabla v)|_{\partial K} \in L^p(\partial K)\,{}^\forall K \in \mathcal{T}_h\}$
$V^p(\Omega_0)$: restriction of V^p to Ω_0, $\mathring{V}^p(\Omega_0) := \{v \in V^p(\Omega_0) \colon \operatorname{supp} v \subset \Omega_0\}$
$V_h = V_h^r(\Omega) := \{v_h \in L^2(\Omega) \colon v_h|_K$ is polynomial of degree $\leq r\,{}^\forall K \in \mathcal{T}_h\}$
$V_h(\Omega_0)$, $\mathring{V}_h(\Omega_0)$: defined similarly.

Notations

$\mathcal{E}_h$: the set of all edges of $K \in \mathcal{T}_h$
$\mathcal{E}_h^\partial$ (resp. $\mathcal{E}_h^\circ$): the set of all edges on boundary (resp. in interior) of Ω
For $v \in V^p$ and $e \in \mathcal{E}_h$, define $\{\!\{\cdot\}\!\}$ and $[\![\cdot]\!]$ as below.

- If $e \in \mathcal{E}_h^\circ$,

$$\{\!\{v\}\!\} := \frac{1}{2}(v_1 + v_2), [\![v]\!] := v_1 n_1 + v_2 n_2,$$
$$\{\!\{\nabla v\}\!\} := \frac{1}{2}(\nabla v_1 + \nabla v_2), [\![\nabla v]\!] := \nabla v_1 \cdot n_1 + \nabla v_2 \cdot n_2.$$

- If $e \in \mathcal{E}_h^\partial$,

$$\{\!\{v\}\!\} := v,\ [\![v]\!] := vn,\ \{\!\{\nabla v\}\!\} := \nabla v,\ [\![\nabla v]\!] := \nabla v \cdot n.$$

$v_i = v|_{K_i}$ (For $e \in \mathcal{E}_h^\circ$, K_1 and K_2 satisfy $e \subset \partial K_1 \cap \partial K_2$)
n_i: outer unit normal vector of e on K_i
n: outer unit normal vector of e on $\partial\Omega$

Norms

Norm of $V^p(\Omega)$

- For $1 \le p < \infty$,

$$\|v\|^p_{V^p(\Omega_0)} := \sum_{K\in\mathcal{T}_h} \|v\|^p_{W^{1,p}(K\cap\Omega_0)} + \sum_{e\in\mathcal{E}_h} h_e^{1-p} \|[\![v]\!]\|^p_{L^p(e\cap\overline{\Omega}_0)} + \sum_{e\in\mathcal{E}_h} h_e \|\{\!\{\nabla v\}\!\}\|^p_{L^p(e\cap\overline{\Omega}_0)}.$$

- For $p = \infty$,

$$\|v\|_{V^\infty(\Omega_0)} := \max_{K\in\mathcal{T}_h} \|v\|_{W^{1,\infty}(K\cap\Omega_0)} + \max_{e\in\mathcal{E}_h} h_e^{-1} \|[\![v]\!]\|_{L^\infty(e\cap\overline{\Omega}_0)} + \max_{e\in\mathcal{E}_h} \|\{\!\{\nabla v\}\!\}\|_{L^\infty(e\cap\overline{\Omega}_0)}.$$

Here, $h_e = (h_{K_1} + h_{K_2})/2$ if $e \in \mathcal{E}_h^\circ$ and $h_e = h_K$ if $e \in \mathcal{E}_h^\partial$.

Scheme of DG

Bilinear form and functional

For $u \in V^p$ and $v \in V^{p'}$,

$$a(u,v) := \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v dx - \sum_{e \in \mathcal{E}_h} \int_e (\{\!\{\nabla u\}\!\} [\![v]\!] + \{\!\{\nabla v\}\!\} [\![u]\!]) ds + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \int_e [\![u]\!] [\![v]\!] ds,$$

$$F(v) := \int_\Omega f v dx + \sum_{e \in \mathcal{E}_h^\partial} \int_e g \left(\frac{\sigma}{h_e} v - \nabla v \cdot n \right) ds.$$

σ: sufficiently large constant

Scheme of DG

$$\text{Find} \quad u_h \in V_h \quad \text{s.t.} \qquad (4)$$
$$a(u_h, \chi) = F(\chi) \quad {}^\forall \chi \in V_h$$

Prop. 2 (Consistency)

The solution u of (1) satisfies $u \in H^s(\Omega)$ for some $s > \frac{3}{2}$. Then,

$$a(u, v) = F(v) \quad {}^\forall v \in V^2. \qquad (5)$$

In particular, if $u_h \in V_h$ is a solution of (4) then Galerkin orthogonality

$$a(u - u_h, \chi) = 0 \quad {}^\forall \chi \in V_h. \qquad (6)$$

3 . Main theorems

Assumptions

PDE on disk D

$$\begin{cases} -\Delta u + u = f & \text{in} \quad D \\ \partial_n u = 0 & \text{on} \quad \partial D \end{cases} \tag{7}$$

$D \subset\subset \Omega$: open disk of center at x_0 and radius R
∂_n: normal derivative on ∂D

Bilinear form of (7):

$$a_D^1(u,v) := \sum_{K \in \mathcal{T}_h} \int_{K \cap D} (\nabla u \cdot \nabla v + uv) dx$$
$$- \sum_{e \in \mathcal{E}_h} \int_{e \cap \overline{D}} (\{\!\{\nabla u\}\!\} [\![v]\!] + \{\!\{\nabla v\}\!\} [\![u]\!]) ds + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \int_{e \cap \overline{D}} [\![u]\!] [\![v]\!] ds$$

Assumption 1

There exists a function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ which is bounded neighborhood of 0 such that

$$a_D^1(u - u_h, v) \le Ch\alpha(h) \|f\|_{L^2(D)} \|v\|_{V^\infty(D)} \qquad {}^\forall f \in L^2(D),\ {}^\forall v \in V^\infty(D) \tag{8}$$

for sufficiently small h.
C is a positive constant independent of h.
The solution u of (7) and u_h satisfy Galerkin orthogonality for a_D^1.

Norm $\|\cdot\|_{*,\Omega_0}$

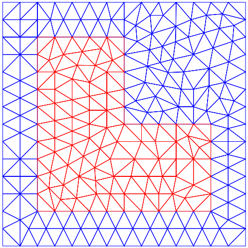
$$\|v\|_{*,\Omega_0} := \|v\|_{L^\infty(\Omega_0)} + \alpha(h) \|v\|_{V^\infty(\Omega_0)}$$

We can take $\alpha(h) = 1$ by properties of a_D^1.

Assumption 2

There exist convex polygonal domain $\widetilde{\Omega} \supset\supset \Omega$ and its triangulation $\widetilde{\mathcal{T}}_h$ such that the restriction of $\widetilde{\mathcal{T}}_h$ into Ω is $\mathcal{T}_h$, and the constant C in (3) is same.

We define $\widetilde{\mathcal{E}}_h$, $V^p(\widetilde{\Omega})$, $V_h(\widetilde{\Omega})$ using $\widetilde{\mathcal{T}}_h$.



Ω(Red) and $\widetilde{\Omega}$(Blue)

Interior error estimate

Thm. 1 (Interior error estimate)

Let κ be a positive constant and open sets $\Omega_0 \subset \Omega_1 \subset \Omega$ satisfy $d = d(\Omega_0, \Omega_1) \geq \kappa h$. If $u \in V^\infty$ and $u_h \in V_h$satisfy

$$a(u - u_h, \chi) = 0 \quad {}^\forall \chi \in \mathring{V}_h.$$

Then, under the assumption 1, there exists a positive constant C independent of h, u, andu_h such that

$$\|u - u_h\|_{L^\infty(\Omega_0)} \leq C \left(\inf_{\chi \in V_h} \|u - \chi\|_{*,\Omega_1} + \|u - u_h\|_{L^2(\Omega_1)} \right) \qquad \text{(A)}$$

for sufficiently small h.

Weak discrete maximum principle

Thm. 2 (Weak discrete maximum principle)

Assume that $u_h \in V_h$ is discrete harmonic, i.e. u_h satisfies

$$a(u_h, \chi) = 0 \quad {}^{\forall}\chi \in \mathring{V}_h. \tag{9}$$

Then, under the assumption 1, there exists a positive constant C independent of h and u_h, such that

$$\|u_h\|_{L^\infty(\Omega)} \leq C \|u_h\|_{L^\infty(\partial\Omega)} \tag{B}$$

for sufficiently small h.

L^∞ error estimate

Thm. 3 (L^∞ error estimate)

If $u \in V^\infty$ and $u_h \in V_h$ satisfy

$$a(u - u_h, \chi) = 0 \quad {}^\forall \chi \in V_h.$$

Then, under the assumption 1 and 2, there exists a positive constant C independent of h, u, and u_h such that

$$\|u - u_h\|_{L^\infty(\Omega)} \le C \left(\inf_{\chi \in V_h} \|u - \chi\|_{*,\Omega} + \|u - u_h\|_{L^\infty(\partial\Omega)} \right) \qquad \text{(C)}$$

for sufficiently small h.

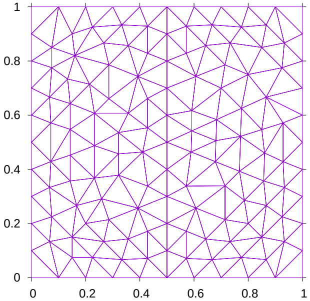
Cor. 1

Under the assumption of 3, if $u \in W^{r+1,\infty}(\Omega)$, then,

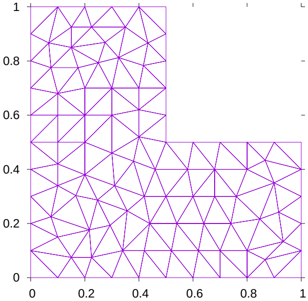
$$\|u - u_h\|_{L^\infty(\Omega)} \le C h^{r-1} \|u\|_{W^{r+1,\infty}(\Omega)}. \qquad \text{(D)}$$

4 . Numerical results

We perform numerical computations for Thm. 2 and Cor. 1 in two domains below.



Square domain



L-shaped domain

Numerical results for Thm. 2

For $f = 0$, $g = \cos(\pi x)\cos(\pi y)$, the numerical solution u_h satisfies (9). Maximum and minimum of u_h in Ω and $\partial\Omega$ are below.

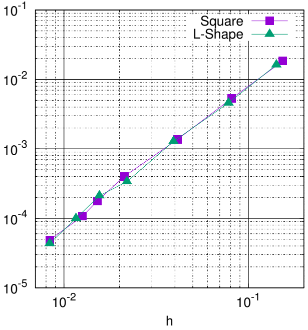
Shape	h	$\min_{\Omega} u_h$	$\min_{\partial\Omega} u_h$	$\max_{\Omega} u_h$	$\max_{\partial\Omega} u_h$
Square	$0.152\ldots$	$-1.014\ldots$	$-1.014\ldots$	$1.014\ldots$	$1.014\ldots$
	$0.076\ldots$	$-1.004\ldots$	$-1.004\ldots$	$1.004\ldots$	$1.004\ldots$
L-shaped	$0.152\ldots$	$-1.014\ldots$	$-1.014\ldots$	$1.014\ldots$	$1.014\ldots$
	$0.079\ldots$	$-1.004\ldots$	$-1.004\ldots$	$1.004\ldots$	$1.004\ldots$

Maximum and minimum of u_h in Ω and $\partial\Omega$

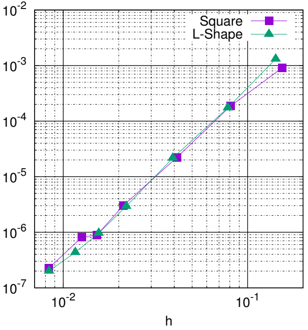
The constant C in (B) is near by 1. We expect it satisfies discrete maximum principle.

Numerical results of Cor. 2

We calculate L^∞ error between exact solution $u(x,y) = \sin(\pi x)\sin(\pi y)$ and numerical solution u_h of Poisson equation.
Orders of error are about $O(h^{r+1})$. It is better result than Cor .2.



L^∞ error for $r = 1$

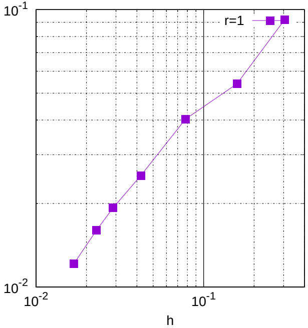


L^∞ error for $r = 2$

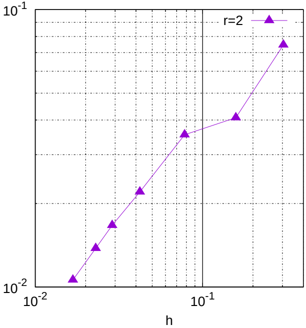
Error of FEM in [Sch80] is $O(h^2|\log h|)$ if $r = 1$ and $O(h^{r+1})$ if $r \geq 2$.

Numerical results of Cor. 2

In L-shaped domain, we calculate error for u which has the correct singular asymptotic of $r^{2/3}$ near the reentrant corner.
Errors decrease according to decreasing of h. But orders are less than $O(h)$.



L^∞ Error for $r = 1$



L^∞ Error for $r = 2$

5 . Brief proofs

Brief proof

Lem. 1

Let $\tilde{u} \in V^{\infty}(D)$ satisfying $\operatorname{supp} \tilde{u} \subset \frac{1}{2} D$. If $\tilde{u}_h \in V_h(D)$ satisfies

$$a_D^1(\tilde{u} - \tilde{u}_h, \chi) = 0 \quad {}^{\forall}\chi \in V_h(D).$$

Then,

$$\|\tilde{u} - \tilde{u}_h\|_{L^{\infty}(\frac{1}{4}D)} \leq C \|\tilde{u}\|_{*,D} \tag{10}$$

for sufficiently small h.

Lem. 2

If w_h satisfies $a_D^1(w_h, \chi) = 0 \quad {}^{\forall}\chi \in \mathring{V}_h(D)$. Then,

$$|w_h(x_0)| \leq C \|w_h\|_{L^2(D)} \tag{11}$$

for sufficiently small h.

Brief proof of Thm. 1

Thm. 1

$$\|u - u_h\|_{L^\infty(\Omega_0)} \le C \left(\inf_{\chi \in V_h} \|u - \chi\|_{*,\Omega_1} + \|u - u_h\|_{L^2(\Omega_1)} \right) \qquad \text{(A)}$$

Brief proof.

Firstly, we show similar estimate for $a^1(u,v) := a(u,v) + (u,v)_\Omega$.
Let $R < d(\Omega_0, \Omega_1)$ and $D \subset \Omega_1$ be an open disk centered at $x_0 = \mathrm{argmax}_{x \in \Omega_0} |u(x) - u_h(x)|$ and radius R.
We take $\omega \in C_0^\infty(\frac{1}{2}D)$ as $0 \le \omega \le 1$ and $\omega \equiv 1$ on $\frac{1}{4}D$, and put $\tilde{u} := \omega u$.
Let $\tilde{u}_h$ satisfy assumption of Lem. 1. Then $\tilde{u}_h - u_h$ satisfies the assumption of Lem. 2 on $\frac{1}{4}D$ and

$$|\tilde{u}_h(x_0) - u_h(x_0)| \le C \|\tilde{u}\|_{*,\frac{1}{2}D} + C \|u - u_h\|_{L^2(D)}.$$

We get the estimate for a^1 and prove Thm. 1 using the estimate of a^1.

Brief proof of Thm. 2

Thm. 2

$$\|u_h\|_{L^\infty(\Omega)} \leq C \|u_h\|_{L^\infty(\partial\Omega)} \tag{B}$$

Brief proof. Step 1.

Let $x_0 \in \Omega$ satisfy $|u_h(x_0)| = \|u_h\|_{L^\infty(\Omega)}$ and put $d = \mathrm{dist}(x_0, \partial\Omega)$, $\rho = \max\{d, h\}$. By Thm. 1 and inverse inequality,

$$\|u_h\|_{L^\infty(\Omega)} \leq C\rho^{-1} \|u_h\|_{L^2(S_\rho(x_0))}. \tag{12}$$

Let $\phi \in C_0^\infty(S_\rho(x_0))$ satisfy $\|\phi\|_{L^2(S_\rho(x_0))} = 1$. Let $v \in H_0^1(\Omega)$ be the solution of (1) for $f = \phi$ and $g = 0$.
Then, $v \in W^{2,p}(\Omega)$ for some $4/3 < p \leq 2$ and $a(v, w) = (\phi, w)_\Omega$ for each $w \in V^2$. Let $v_h \in \mathring{V}_h$ satisfy

$$a(v - v_h, \chi) = 0 \quad {}^\forall \chi \in \mathring{V}_h.$$

Brief proof of Thm. 2

Brief proof. Step 2.

By assumption of u_h and properties of a,

$$|(u_h, \phi)_\Omega| \le Ch^{-1} \|u_h\|_{L^\infty(\partial\Omega)} \|v - v_h\|_{V^1(\Lambda_h)} \tag{13}$$

where $\Lambda_h := \{x \in \overline{\Omega} \colon \operatorname{dist}(x, \partial\Omega) \le h\}$.
We put $d_j = R_0 2^{-j}$ and divide Ω into annuli $A_j := \{x \in \overline{\Omega} \colon d_{j+1} \le |x - x_0| \le d_j\}$. By L^2 error estimate, properties of Poisson equation, $h \le \rho \le Cd_j \le R_0$, and $p > 4/3$,

$$\begin{aligned} \|v - v_h\|_{V^1(\Lambda_h)} &\le C \sum_{j=0}^{J} h^{1/2} d_j^{1/2} \|v - v_h\|_{V^2(\Lambda_h \cap A_j)} \\ &\quad + C\rho^{1/2} h^{1/2} \|v - v_h\|_{V^2(\Lambda_h \cap S_{8\rho}(x_0))} \\ &\le Ch\rho. \end{aligned} \tag{14}$$

Using (12), (13) and (14), (B) holds.

Brief proof of Thm. 3

Thm. 3

$$\|u-u_h\|_{L^\infty(\Omega)} \le C\left(\inf_{\chi\in V_h}\|u-\chi\|_{*,\Omega} + \|u-u_h\|_{L^\infty(\partial\Omega)}\right) \qquad \text{(C)}$$

Brief proof. Step 1.

Let $\tilde{u}\in V^\infty(\widetilde{\Omega})$ be the extension of u and satisfy $\|\tilde{u}\|_{V^\infty(\widetilde{\Omega})} \le C\|u\|_{V^\infty(\Omega)}$ and $\tilde{u}=0$ over $\partial\widetilde{\Omega}$. Let $\tilde{u}_h \in \mathring{V}_h(\widetilde{\Omega})$ satisfy

$$\tilde{a}(\tilde{u}-\tilde{u}_h,\chi)=0 \quad {}^\forall\chi\in\mathring{V}_h(\widetilde{\Omega}).$$

Here, $\tilde{a}$ is defined by similarly of a using $\widetilde{\mathcal{T}}_h$ and $\widetilde{\mathcal{E}}_h$. By Thm. 1,

$$\|\tilde{u}-\tilde{u}_h\|_{L^\infty(\Omega)} \le C\|\tilde{u}\|_{*,\widetilde{\Omega}} + C\|\tilde{u}-\tilde{u}_h\|_{L^2(\widetilde{\Omega})}. \qquad (15)$$

Brief proof of Thm. 3

Brief proof. Step 2.

By continuity of $\tilde{a}$ and regularity of Poisson equation.

$$\|\tilde{u} - \tilde{u}_h\|_{L^2(\widetilde{\Omega})} \leq Ch \|\tilde{u}\|_{V^\infty(\widetilde{\Omega})} . \tag{16}$$

Since $a(u_h - \tilde{u}_h, \chi) = 0$ for $\chi \in \mathring{V}_h$,

$$\begin{aligned} \|u_h - \tilde{u}_h\|_{L^\infty(\Omega)} &\leq C \|u_h - \tilde{u}_h\|_{L^\infty(\partial\Omega)} \\ &\leq C \|\tilde{u}\|_{*,\widetilde{\Omega}} + C \|u - u_h\|_{L^\infty(\partial\Omega)} \end{aligned} \tag{17}$$

using Thm. 2, 1 and (16).
Since $u - u_h = (u - \chi) - (u_h - \chi)$ for each $\chi \in V_h$, (C) holds.

Conclusions

- We get interior error estimate, weak discrete maximum principle, and L^∞ error of DG method for Poisson equation.
- However, the result of estimate is lough comparing with [CC04] and numerical result.
- For future work, we will analyze $\alpha(h)$ and $\|u - u_h\|_{L^\infty(\partial\Omega)}$ more strictly, and conclude the estimate like FEM in [Sch80].

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