Discontinuous Galerkin method for an N-dimensional spherically symmetric Poisson equation

Yuki Chiba Norikazu Saito

Graduate School of Mathematical Sciences, the University of Tokyo

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Contents

1 . Introduction

Introduction

In the theory of PDE, it is known several critical phenomena with critical value related to space dimension N.

Example: Positive solution of semilinear elliptic equation

$N \geq 3$, $\Omega \subset \mathbb{R}^N$: smooth bounded domain.

$$\begin{cases} \Delta u + |u|^{p-1} u = 0 & (x \in \Omega) \\ u = 0 & (x \in \partial\Omega) \end{cases}$$

If $1 < p < p_s = (N+2)/(N-2)$, positive solution exists. If $p > p_s$, positive solution may not exist.

If we can make numerical method for nonlinear PDEs in higher dimension, it is useful for studying critical phenomena through experimental consideration.

Model Problem

Consider the Poisson equation in N-dimensional ball
$B_R = \{\xi \in \mathbb{R}^N \mid |\xi|_{\mathbb{R}^N} < R\}$.

Poisson equation

$$\begin{cases} -\Delta_\xi U(\xi) + Q(\xi)U(\xi) = F(\xi) & (\xi \in B_R) \\ U(\xi) = 0 & (\xi \in \partial B_R) \end{cases} \tag{1}$$

Set $x = |\xi|$ and we assume that coefficient function Q and F are spherically symmetric $Q(\xi) = \hat{q}(x)$, $F(\xi) = \hat{f}(x)$.
Then, (1) is reduced to next equation.

$$\begin{cases} -\frac{1}{x^{N-1}} \left(x^{N-1}\hat{u}_x\right)_x + \hat{q}\hat{u} = \hat{f} & (x \in I = (0, R)) \\ \hat{u}_x(0) = \hat{u}(R) = 0 \end{cases} \tag{2}$$

Model Problem

In previous study, there are two FEMs using weight function to eliminate singularity. (cf. K, Ericsson and V, Thomée. 1984)

1. Using weight function x^{N-1}

$$-(x^{N-1}\hat{u}_x)_x + x^{N-1}\hat{q}\hat{u} = x^{N-1}\hat{f}$$

2. Using weight function x

$$-(x\hat{u}_x)_x + (2-N)\hat{u}_x + x\hat{q}\hat{u} = x\hat{f}$$

In this study, we apply Discontinuous Galerkin (DG) method to second case.

2 . DG Scheme

DG Scheme

Generally, consider the following problem.

Diffusion-convection equation

$$\begin{cases} -(\nu u_x)_x + b u_x + q u = f \quad (x \in I) \\ u_x(0) = u(R) = 0 \end{cases} \tag{3}$$

$\nu(x) = x$, $b \leq 0$:Const., q, $f \in L^2(I)$, $q(x) > 0$ $(x \in I)$

Divition $\mathcal{T}_h = \{K_i\}_{i \in \Lambda}$ of I is defined by below.

$$0 = x_1 < x_2 < \cdots < x_i < \cdots < x_n = R$$

$$K_i = (x_i, x_{i+1}), \quad h_i = |K_i| = x_{i+1} - x_i,$$

$$h = \max_{i \in \Lambda} h_i, \quad \Lambda = \{1, 2, \ldots, n-1\}$$

$$e_i = \min\{h_i, h_{i-1}\} \, (i = 2, \ldots, n-1), \quad e_n = h_{n-1}$$

DG Scheme

Function space

$$H^m(\mathcal{T}_h) = \{v \in L^2(I) \mid v|_{K_i} \in H^m(K_i)\ (i \in \Lambda)\}$$
$$V_h = V_h^k = \{v \in L^2(I) \mid v|_{K_i} \in \mathcal{P}^k(K_i)\ (i \in \Lambda)\}$$

Notation: For $v \in H^1(\mathcal{T}_h)$, $v^i = v|_{K_i}\ (i \in \Lambda)$

$$\nu_i = \nu(x_i) = x_i, \qquad (u, v)_i = \int_{x_i}^{x_{i+1}} uv\ dx$$

$$[\![v]\!]_i = \begin{cases} -v^1(x_1) & (i = 1) \\ v^{i-1}(x_i) - v^i(x_i) & (2 \le i \le n-1) \\ v^{n-1}(x_n) & (i = n) \end{cases}$$

$$\langle\!\langle v \rangle\!\rangle_i = \begin{cases} v^1(x_1) & (i = 1) \\ \frac{v^{i-1}(x_i) + v^i(x_i)}{2} & (2 \le i \le n-1) \\ v^{n-1}(x_n) & (i = n) \end{cases}$$

DG Scheme

$$\begin{aligned}&\text{Find} \quad u_h \in V_h \quad \text{s.t.}\\ &a_h(u_h, v) = a_h^{\mathrm{d}}(u_h, v) + a_h^{\mathrm{cr}}(u_h, v) = (f, v) \quad ({}^\forall v \in V_h)\end{aligned} \tag{4}$$

$$\begin{aligned}a_h^{\mathrm{d}}(u, v) &= \sum_{i=1}^{n-1} (\nu u_x, v_x)_i - \sum_{i=2}^{n} \nu_i \langle\!\langle u_x \rangle\!\rangle_i [\![v]\!]_i \\ &\qquad - \alpha \sum_{i=2}^{n} \nu_i \langle\!\langle v_x \rangle\!\rangle_i [\![u]\!]_i + \sum_{i=2}^{n} \frac{\nu_i \sigma}{e_i} [\![u]\!]_i [\![v]\!]_i \\ a_h^{\mathrm{cr}}(u, v) &= -\sum_{i=1}^{n-1} (bu, v_x)_i + \sum_{i=1}^{n-1} b \langle\!\langle u \rangle\!\rangle_i [\![v]\!]_i + \sum_{i=2}^{n-1} \frac{1}{2} |b| [\![u]\!]_i [\![v]\!]_i + \sum_{i=1}^{n-1} (qu, v)_i \\ (f, v) &= \sum_{i=1}^{n-1} (f, v)_i\end{aligned}$$

3 . Analysis of Scheme

Norms

We introduce following DG norms.

$$\|v\|_{\mathrm{d}}^2 = \sum_{i=1}^{n-1} (\nu v_x, v_x)_i + \sum_{i=2}^{n} \frac{\nu_i \sigma}{e_i} [\![v]\!]_i^2, \qquad \|v\|_{\mathrm{d},*}^2 = \|v\|_{\mathrm{d}}^2 + \sum_{i=1}^{n-1} h_i^2 (\nu v_{xx}, v_{xx})_i$$

$$\|v\|_{\mathrm{cr}}^2 = \sum_{i=1}^{n-1} (qv, v)_i + \sum_{i=1}^{n} \frac{1}{2} |b| [\![v]\!]_i^2, \qquad \|v\|_{\mathrm{cr},*}^2 = \|v\|_{\mathrm{cr}}^2 + \sum_{i=1}^{n-1} |b| \langle\!\langle v \rangle\!\rangle_i^2$$

$$|\!|\!| v |\!|\!|^2 = \|v\|_{\mathrm{d}}^2 + \|v\|_{\mathrm{cr}}^2, \qquad |\!|\!| v |\!|\!|_*^2 = \|v\|_{\mathrm{d},*}^2 + \|v\|_{\mathrm{cr},*}^2$$

Assume that division $\{\mathcal{T}_h\}_h$ is quasi-uniform.

$$^{\exists}\theta_0 > 0 \quad \text{s.t.} \quad 0 < \frac{h_i}{h_j} \leq \theta_0 \quad (1 \leq {}^{\forall} i, j \leq n, \ {}^{\forall}\mathcal{T}_h \in \{\mathcal{T}_h\}_h) \qquad \text{(A1)}$$

Inequality with Weighted norm

Lem 1 (Trace inequality)

There exists a positive constant $C_1 = C_1(\theta_0)$ satisfying

$$\nu_i(v_x^i(x_i))^2 \le C_1(h_i^{-1}(\nu v_x, v_x)_i + h_i(\nu v_{xx}, v_{xx})_i)$$

$$\nu_i(v_x^i(x_{i+1}))^2 \le C_1(h_i^{-1}(\nu v_x, v_x)_i + h_i(\nu v_{xx}, v_{xx})_i).$$

Lem 2 (Inverse inequality)

Let $K = (s,t) \subset \mathbb{R}_{>0}$ be interval and set $\rho = t - s > 0$. Then, There exists a positive constant $C_2 = C_2(k)$ satisfying

$$\int_K x v_{xx}^2 \, dx \le C_2 \rho^{-2} \int_K x v_x^2 \, dx \quad (v \in \mathcal{P}^k).$$

Analysis of Diffusion Term a_h^{d}

Lem 3 (Continuity and coercivity of a_h^{d})

(i) For all $\alpha \in \mathbb{R}$ and $\sigma > 0$, there exists a positive constant $C_{\mathrm{d}} > 0$ independent of h satisfying

$$a_h^{\mathrm{d}}(u, v) \leq C_{\mathrm{d}} \|u\|_{\mathrm{d},*} \|v\|_{\mathrm{d}} \quad (u \in H^2(\mathcal{T}_h),\ v \in V_h).$$

(ii) There exists a positive constant $\sigma_* > 0$ independent of h satisfying below. If $\sigma \geq \sigma_*$, then

$$a_h^{\mathrm{d}}(v, v) \geq \frac{1}{2} \|v\|_{\mathrm{d}}^2 \qquad (v \in V_h).$$

Analysis of Diffusion Term a_h^{d}

Proof

(i) By Cauchy-Schwarz's inequality,

$$|a_h^{\mathrm{d}}(u,v)| \leq \left(\sum_{i=1}^{n-1} (\nu u_x, u_x)_i^2 + \sum_{i=2}^{n} \frac{\nu_i e_i}{\sigma} \langle\!\langle u_x \rangle\!\rangle_i^2 + (1+|\alpha|) \sum_{i=2}^{n} \frac{\nu_i \sigma}{e_i} [\![u]\!]_i^2 \right)^{1/2}$$

$$\cdot \left(\sum_{i=1}^{n-1} (\nu v_x, v_x)_i^2 + \sum_{i=2}^{n} \frac{\nu_i e_i}{\sigma} \langle\!\langle v_x \rangle\!\rangle_i^2 + (1+|\alpha|) \sum_{i=2}^{n} \frac{\nu_i \sigma}{e_i} [\![v]\!]_i^2 \right)^{1/2}.$$

Using trace inequality and inverse inequality, we get the estimate.

Analysis of diffusion term a_h^{d}

Proof

(ii) Using Cauchy-Schwarz's inequality, trace inequality and inverse inequality, for $\delta > 0$,

$$a_h^{\mathrm{d}}(v,v) \geq \left(1 - \frac{1+|\alpha|}{2\delta\sigma} C_1(1+C_2)\right) \sum_{i=1}^{n-1} (\nu v_x, v_x)_i + \left(1 - \frac{1+|\alpha|}{2}\delta\right) \sum_{i=2}^{n} \frac{\nu_i \sigma}{e_i} [\![v]\!]_i^2.$$

Choosing $\delta = 1/(1+|\alpha|)$ and $\sigma \geq \sigma_* = C_1(1+C_2)(1+|\alpha|)/\delta$, we get the estimate.

Analysis of Convection Term a_h^{cr}

For each $i \in \Lambda$, we define $P_{K_i} : L^1(K_i) \to \mathcal{P}^k(K_i)$ as locally L^2 projection operator.
We define globally L^2 projection operator $P_h : L^1(I) \to V_h$ by $(P_h v)|_{K_i} = P_{K_i} v$ $(i \in \Lambda)$.

Lem 4 (Continuity and covercivity of a_h^{cr})

(i) There exists a positive constant $C_{\mathrm{cr}} > 0$ independent of h satisfying

$$a_h^{\mathrm{cr}}(u - P_h u, v) \le C_{\mathrm{cr}} \|u - P_h u\|_{\mathrm{cr},*} \|v\|_{\mathrm{cr}} \quad (u \in H^2(\mathcal{T}_h),\ v \in V_h).$$

(ii) Following inequality holds.

$$a_h^{\mathrm{cr}}(v, v) \ge \frac{1}{2} \|v\|_{\mathrm{cr}}^2 \quad (v \in V_h)$$

Analysis of Convection Term a_h^{cr}

Proof

(i) Set $\phi = u - P_h u$. Since $v_x \in \mathcal{P}^{k-1}(K_i)$, $(\phi, v_x)_i = 0$. Using Cauchy-Schwarz's inequality,

$$
\begin{aligned}
a_h^{\mathrm{cr}}(\phi, v) &\leq \left(\sum_{i=1}^{n-1} |b| \langle\!\langle \phi \rangle\!\rangle_i^2 + \sum_{i=2}^{n-1} \frac{1}{2} |b| [\![\phi]\!]_i^2 + \sum_{i=1}^{n-1} (q\phi, \phi)_i \right)^{1/2} \\
&\quad \cdot \left(\sum_{i=1}^{n-1} |b| [\![v]\!]_i^2 + \sum_{i=2}^{n-1} \frac{1}{2} |b| [\![v]\!]_i^2 + \sum_{i=1}^{n-1} (qv, v)_i \right)^{1/2} \\
&\leq 3 \|\phi\|_{\mathrm{cr},*} \|v\|_{\mathrm{cr}}.
\end{aligned}
$$

Proof

(ii) Using integration by parts,

$$
\begin{aligned}
-\sum_{i=1}^{n-1}(bv, v_x)_i &= \sum_{i=1}^{n-1}\left[(bv_x, v)_i - b(v^i(x_{i+1})^2 - v^i(x_i)^2)\right] \\
&= \sum_{i=1}^{n-1}(bv_x, v)_i + bv^1(x_1)^2 - bv^{n-1}(x_n)^2 - 2\sum_{i=2}^{n-1} b\langle\!\langle u\rangle\!\rangle_i [\![v]\!]_i.
\end{aligned}
$$

So,

$$
-\sum_{i=1}^{n-1}(bv, v_x)_i = \left[bv^1(x_1)^2 - bv^{n-1}(x_n)^2\right]/2 - \sum_{i=2}^{n-1} b\langle\!\langle u\rangle\!\rangle_i [\![v]\!]_i.
$$

Substituting this for $a_h^{\mathrm{cr}}(v, v)$, we get the estimate.

Analysis of a_h

Lem 5 (Continuity and Coercivity of a_h)

(i) For all $\alpha \in \mathbb{R}$ and $\sigma > 0$, there exists a positive constant $C_{\mathrm{dcr}} > 0$ independent of h satisfying

$$a_h(u - P_h u, v) \le C_{\mathrm{dcr}} |||u - P_h u|||_* |||v||| \quad (u \in H^2(\mathcal{T}_h),\ v \in V_h).$$

(ii) There exists a positive constant σ_* independent of h satisfying below. If $\sigma \ge \sigma_*$, then

$$a_h(v, v) \ge \frac{1}{2} |||v|||^2 \quad (v \in V_h).$$

Analysis of a_h

Thm 1

Let $u \in H^2(I)$ be the solution of (3). Assume that $\sigma \geq \sigma_*$.
Then, there exists a unique solution $u_h \in V_h$ of DG Scheme (4), and it satisfies Galerkin orthogonality

$$a_h(u - u_h, v) = 0 \quad (v \in V_h).$$

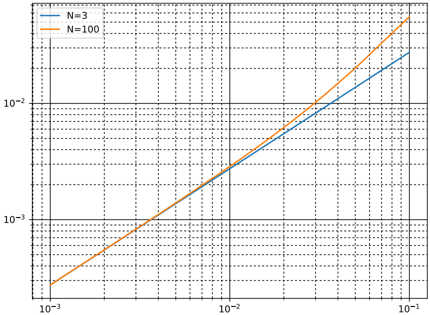
In addition, the following estimate holds.

$$|||u - u_h||| \leq (1 + 2C_{\mathrm{dcr}})\, |||u - P_h u|||_* . \tag{5}$$

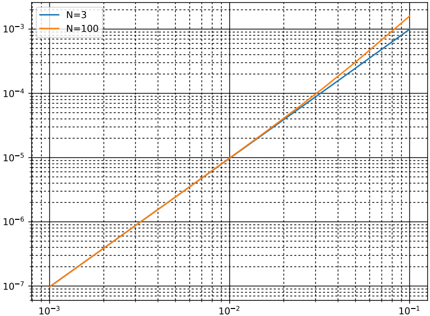
If u is sufficiently smooth, the order of error is $O(h^k)$.

4 . Numerical Results

$I = (0, 1)$, $N = 3, 100$, $b = 2 - N$, $\alpha = 1$, $\sigma = 20$
(i) $u(x) = \cos \frac{\pi}{2} x$



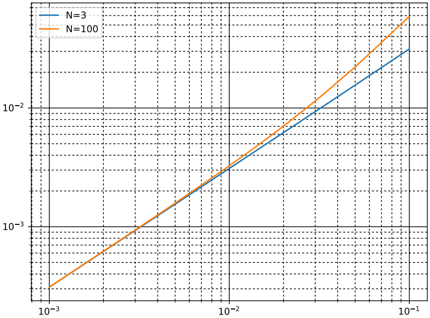
Error of $|||\cdot|||$ using $\mathcal{P}^1$ $(k = 1)$



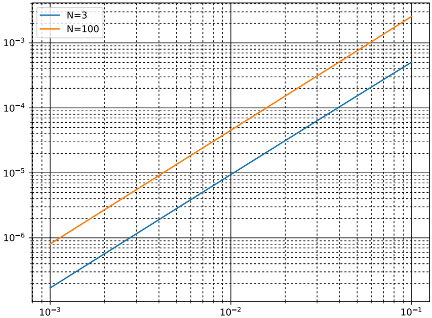
Error of $|||\cdot|||$ using $\mathcal{P}^2$ $(k = 2)$

The order of error is $O(h^k)$ for sufficiently small h.

$I = (0, 1)$, $N = 3, 100$, $b = 2 - N$, $\alpha = 1$, $\sigma = 20$
(ii) $u(x) = x^{7/4} - 1$



Error of $|||\cdot|||$ using $\mathcal{P}^1$ $(k = 1)$

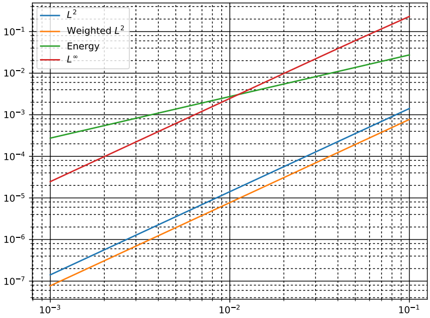


Error of $|||\cdot|||$ using $\mathcal{P}^2$ $(k = 2)$

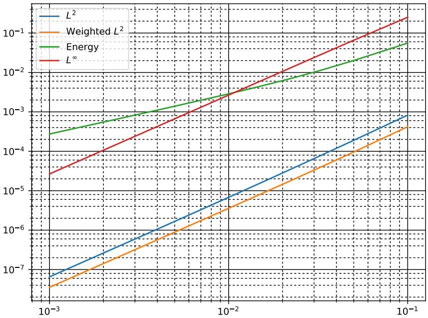
The order of error is $O(h)$ if $k = 1$ and $O(h^{7/4})$ if $k = 2$.

Observations of Other Norms

$I = (0, 1)$, $N = 3, 100$, $b = 2 - N$, $\alpha = 1$, $\sigma = 20$, $\mathcal{P}^1$ element
(i) $u(x) = \cos \frac{\pi}{2} x$



Error for $N = 3$

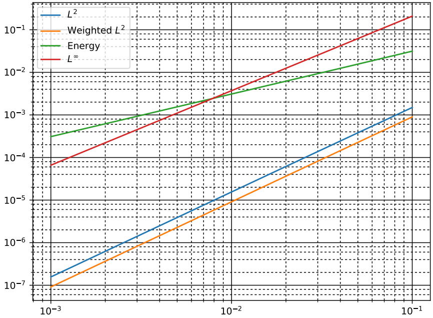


Error for $N = 100$

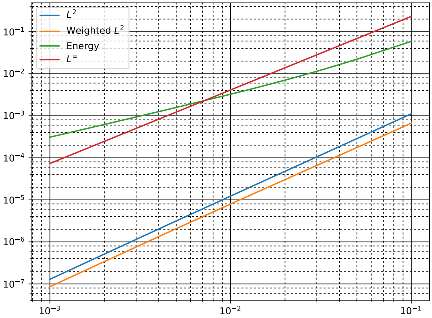
$|||\cdot|||$ norm is $O(h)$, the others are $O(h^2)$.

Observations of Other Norms

$I = (0, 1)$, $N = 3, 100$, $b = 2 - N$, $\alpha = 1$, $\sigma = 20$, $\mathcal{P}^1$ element
(ii) $u(x) = x^{7/4} - 1$



Error for $N = 3$

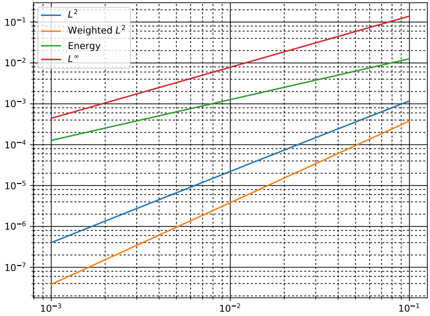


Error for $N = 100$

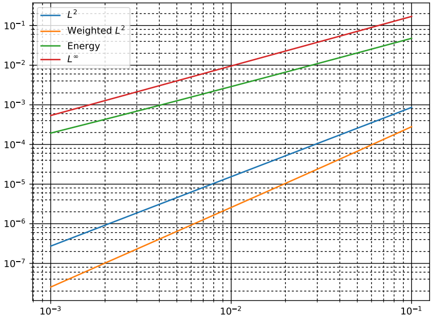
L^2 and weighted L^2 norm are $O(h^2)$, $|||\cdot|||$ norm is $O(h)$, L^∞ norm is $O(h^{7/4})$.

Observations of Other Norms

$I = (0,1)$, $N = 3, 100$, $b = 2 - N$, $\alpha = 1$, $\sigma = 20$, $\mathcal{P}^1$ element
(iii) $u(x) = x^{5/4} - 1$



Error for $N = 3$

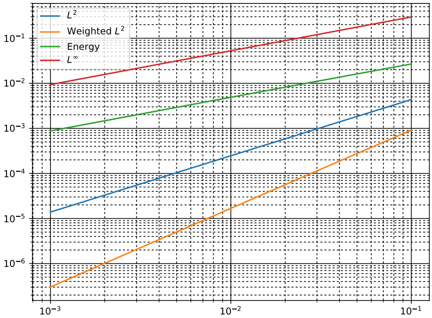


Error for $N = 100$

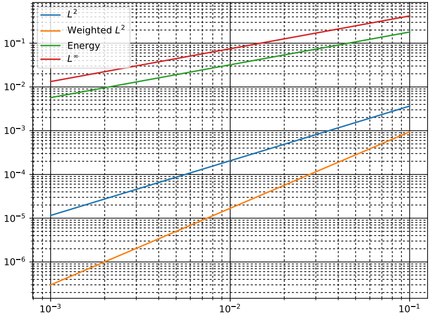
L^2 norm is $O(h^{7/4})$, weighted L^2 norm is $O(h^2)$, $|||\cdot|||$ norm is $O(h)$, L^∞ norm is $O(h^{5/4})$.

Observations of Other Norms

$I = (0, 1)$, $N = 3, 100$, $b = 2 - N$, $\alpha = 1$, $\sigma = 20$, $\mathcal{P}^1$ element
(iv) $u(x) = x^{3/4} - 1$



Error for $N = 3$



Error for $N = 100$

L^2 norm is $O(h^{5/4})$, weighted L^2 norm is $O(h^{7/4})$, $|||\cdot|||$ norm and L^∞ norm are $O(h^{3/4})$.

5 . Modified DG Scheme

Modified DG Scheme

$$\text{Find} \quad u_h \in V_h \quad \text{s.t.}$$
$$b_h(u_h, v) = (f, v) \quad ({}^\forall v \in V_h) \tag{6}$$

$$b_h(u,v) = \sum_{i=1}^{n-1} (\nu u_x, v_x)_i - \sum_{i=2}^{n} \nu_i \langle\!\langle u_x \rangle\!\rangle_i [\![v]\!]_i + \sum_{i=2}^{n} \frac{\nu_i \sigma}{e_i} [\![u]\!]_i [\![v]\!]_i$$
$$+ \sum_{i=1}^{n-1} (b u_x, v)_i - \sum_{i=2}^{n} b [\![u]\!]_i \langle\!\langle v \rangle\!\rangle_i + \sum_{i=2}^{n-1} \frac{1}{2} |b| [\![u]\!]_i [\![v]\!]_i + \sum_{i=1}^{n-1} (qu, v)_i$$

Modified DG Scheme

Thm 2

Let $u \in H^2(I)$ be the solution (3). Assume that $\sigma \geq \sigma_*$.
Then, there exists a unique solution $u_h \in V_h$ of DG Scheme (6), and it satisfies Galerkin orthogonality.
In addition, there exists a positive constant $C > 0$ independent of h satisfying

$$\|u - u_h\|_{L^\infty(I)} \leq C(h \inf_{\chi \in V_h} \|u - \chi\|_{\mathrm{DG},\infty,0} + \max_{i \in \Lambda} \left|(u - u_h)^i(x_{i+1})\right|)$$

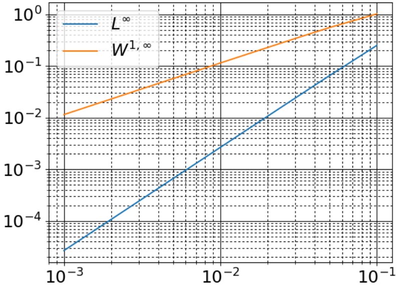
$$\max_{i \in \Lambda} \|(u - u_h)_x\|_{L^\infty(K_i)} \leq C(\inf_{\chi \in V_h} \|u - \chi\|_{\mathrm{DG},\infty,0} + \max_{i \in \Lambda} \left|(u - u_h)^i(x_{i+1})\right|)$$

for sufficiently small h. Moreover, if $q = 0$, we have

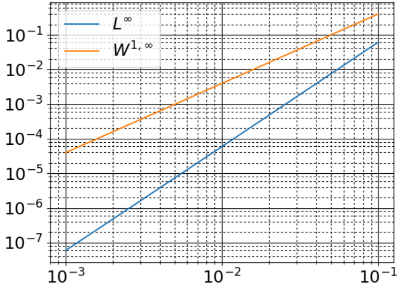
$$\max_{i \in \Lambda} \|(u - u_h)_x\|_{L^\infty(K_i)} \leq C \inf_{\chi \in V_h} \|u - \chi\|_{\mathrm{DG},\infty,0}.$$

Modified DG Scheme -Numerical Result-

$I = (0, 1)$, $N = 100$, $b = 2 - N$, $\sigma = 20$
$u(x) = \cos \frac{\pi}{2} x$



Error using $\mathcal{P}^1$ $(k = 1)$



Error using $\mathcal{P}^2$ $(k = 2)$

L^∞ norm is $O(h^{k+1})$ and piecewize $W^{1,\infty}$ semi norm is $O(h^k)$.

Conclusion

- We have introduced DG schemes for a singular-perturbation elliptic problem derived from a spherical symmetric Poisson equation in the N dimensional ball. We have derived error estimates in the DG energy norm.
- We have confirmed the rate of convergence by numerical experiments. Optimal orders (depending on the regularity of solutions) were actually observed.
- Some point-wise estimates were obtained for a modified DG scheme.
- In the future work, We will apply the results to evolution equations and extend to nonlinear problems.