Nitsche's method and discontinuous Galerkin method for Poisson equation with Robin boundary condition in a smooth domain

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Contents

1 . Introduction

Introduction

Numerical computitaion for a smooth domain

We utilize polyhedral approximations of the domain.
For the standard FEM, the methods and analysis are well developed so far.
For other methods including the DG method, there is room for further study.

Analysis of Nitsche's method and DG method for Poisson equation with Robin boundary condition in a smooth domain.

Related studies

- Barrett & Elliott(1988)
  FEM for Neumann and Robin condition in a smooth domain
- Juntunen & Stenberg(2008)
  Nitsche's method for Neumann and Robin condition in a polygonal domain

2 . Model equation and Schemes

Model equation

Poisson equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} + \dfrac{1}{\varepsilon} u = \dfrac{1}{\varepsilon} u_0 + g & \text{on} \quad \Gamma \end{cases} \tag{1}$$

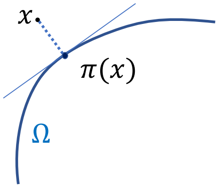
$\Omega \subset \mathbb{R}^d \, (d = 2$ or $3)$, its boundary $\Gamma = \partial\Omega$ is sufficiently smooth.
$0 < \varepsilon < \infty$, ν: outward unit normal vector

$\varepsilon \to 0$: Diriclet condition, $\varepsilon \to \infty$: Neumann condition

Preliminaries-Domain

Ω_h: polygonal domain, $\Gamma_h = \partial\Omega_h$, $\mathcal{T}_h$: shape-regular triangulation of Ω_h
$\overline{\mathcal{I}_h}$: the set of all edges (or faces) of $K \in \mathcal{T}_h$,
$\mathcal{E}_h = \{E \in \overline{\mathcal{I}_h} \colon E \subset \Gamma_h\}$: partition of Γ_h
Assume that every vertices of $E \in \mathcal{E}_h$ lies on Γ.
On neighborhood of Γ, there exists orthogonal projection π satisfying

$$x = \pi(x) + d(x)\nu(\pi(x))$$



where $d(x)$ is signed distance function.

Preliminaries-Domain

Lem. 1 (Boundary-skin estimates(Kashiwabara-Oikawa-Zhou(2016), Kashiwabara-Kemmochi(preprint)))

The following estimates hold.

$$\left| \int_{\pi(E)} f \, d\gamma - \int_E f \circ \pi \, d\gamma_h \right| \le Ch^2 \int_{\pi(E)} |f| \, d\gamma \quad f \in L^1(\pi(E)),\ E \in \mathcal{E}_h \tag{2}$$

$$\|f - f \circ \pi\|_{L^p(\Gamma_h)} \le Ch^{2-2/p} \|f\|_{W^{1,p}(\Gamma(Ch^2))} \quad f \in W^{1,p}(\Gamma(Ch^2)) \tag{3}$$

$$\|f\|_{L^p(\Gamma(Ch^2))} \le C(h^2 \|\nabla f\|_{L^p(\Gamma(Ch^2))} + h^{2/p} \|f\|_{L^p(\Gamma)}) \quad f \in W^{1,p}(\Gamma(Ch^2)) \tag{4}$$

$$\|f\|_{L^p(\Omega_h \setminus \Omega)} \le C(h^2 \|\nabla f\|_{L^p(\Omega_h \setminus \Omega)} + h^{2/p} \|f\|_{L^p(\Gamma_h)}) \quad f \in W^{1,p}(\Omega_h) \tag{5}$$

$$\|\nu_h - \nu \circ \pi\|_{L^\infty(\Gamma_h)} \le Ch \tag{6}$$

ν_h: outward unit normal vector of Γ_h, $\Gamma(Ch^2)$: neighborhood of Γ

$\#$ represents N and DG because the properties of Nitsche's and DG methods are quite similar.
$\widetilde{\Omega} \subset \mathbb{R}^d$: sufficiently smooth domain which inclueds $\Omega, \Omega_h, \Gamma(Ch^2)$.
$\tilde{f}$: extension of f to $\widetilde{\Omega}$.
Assume that u_0, g are the restrictions of functions $\tilde{u}_0$, $\tilde{g}$ defined on $\widetilde{\Omega}$.

P1 finite element spaces V_N, V_{DG}

$$V_N := \{\chi \in C(\overline{\Omega})\colon \chi|_K \in \mathcal{P}^1(K)\ {}^\forall K \in \mathcal{T}_h\}$$
$$V_{DG} := \{\chi \in L^2(\Omega)\colon \chi|_K \in \mathcal{P}^1(K)\ {}^\forall K \in \mathcal{T}_h\}$$

There exist projections $\Pi_\#\colon H^2(\Omega_h) \to V_\#$ satisfying

$$|w - \Pi_\# w|_{H^m(K)} \le Ch^{2-m} \|w\|_{H^2(K)} \quad w \in H^2(K)\, K \in \mathcal{T}_h,\ m = 0, 1, 2.$$

Preliminaries-Numerical flux

$$\mathcal{I}_h = \{E \in \overline{\mathcal{I}_h} \colon \ E \not\subset \Gamma_h\} = \overline{\mathcal{I}_h} \setminus \mathcal{E}_h$$

For $v \in H^s(\Omega_h) + V_{DG}$ and $E \in \mathcal{I}_h$, define $\{\!\{\cdot\}\!\}$ and $[\![\cdot]\!]$ as below.

$$\{\!\{v\}\!\} := \frac{1}{2}(v_1 + v_2)\,,\ [\![v]\!] := v_1 n_1 + v_2 n_2\,,$$
$$\{\!\{\nabla v\}\!\} := \frac{1}{2}(\nabla v_1 + \nabla v_2)\,, [\![\nabla v]\!] := \nabla v_1 \cdot n_1 + \nabla v_2 \cdot n_2$$

$K_1,\, K_2 \in \mathcal{T}_h$: distinct elements satisfying $E = \overline{K}_1 \cap \overline{K}_2$, $v_i = v|_{K_i}$
n_i: outward unit normal vector on E wit respect to K_i

For $E \in \mathcal{E}_h$, $\{\!\{\nabla v\}\!\} = \dfrac{\partial v}{\partial \nu_h}$.

Nitsche's scheme

Nitsche's method

Find $u_N \in V_N$ s.t. $a_h^N(u_N, \chi) = l_h(\chi) \quad {}^\forall \chi \in V_N$ (N)

$$a_h^N(w,v) = \sum_{K \in \mathcal{T}_h} (\nabla w, \nabla v)_K + b_h(w,v)$$

$$b_h(w,v) = \sum_{E \in \mathcal{E}_h} \Big\{ -\frac{\gamma h_E}{\varepsilon + \gamma h_E} \Big(\langle \frac{\partial w}{\partial \nu_h}, v \rangle_E + \langle w, \frac{\partial v}{\partial \nu_h} \rangle_E \Big) + \frac{1}{\varepsilon + \gamma h_E} \langle w, v \rangle_E - \frac{\varepsilon \gamma h_E}{\varepsilon + \gamma h_E} \langle \frac{\partial w}{\partial \nu_h}, \frac{\partial v}{\partial \nu_h} \rangle_E \Big\}$$

$$l_h(v) = (\tilde{f}, v)_{\Omega_h} + \sum_{E \in \mathcal{E}_h} \Big\{ \frac{1}{\varepsilon + \gamma h_E} \langle \tilde{u}_0 + \varepsilon \tilde{g}, v - \gamma h_E \frac{\partial v}{\partial \nu_h} \rangle_E \Big\}$$

$h_E = \mathrm{diam}\, E$, $\gamma > 0$: sufficiently small constant

DG method

$$\text{Find} \quad u_{DG} \in V_{DG} \quad \text{s.t.} \quad a_h^{DG}(u_{DG}, \chi) = l_h(\chi) \quad {}^\forall \chi \in V_{DG} \qquad \text{(DG)}$$

$$a_h^{DG}(w,v) = \sum_{K \in \mathcal{T}_h} (\nabla w, \nabla v)_K + b_h(w,v) + J_h(w,v)$$

$$J_h(w,v) = \sum_{E \in \mathcal{I}_h} \left\{ -\langle \{\!\{ \nabla w \}\!\}, [\![v]\!] \rangle_E - \langle [\![w]\!], \{\!\{ \nabla v \}\!\} \rangle_E + \frac{1}{\gamma h_E} \langle [\![w]\!], [\![v]\!] \rangle_E \right\}$$

3 . Analysis of Schemes

Norms

For $H^s(\Omega_h) + V_{\#}$ $(s > 3/2)$, define some norms by below.

$$\|v\|_N^2 := \|\nabla v\|_{L^2(\Omega_h)}^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{\varepsilon + h_E} \|v\|_{L^2(E)}^2$$

$$\|v\|_{N,h}^2 := \|v\|_N^2 + \sum_{E \in \mathcal{E}_h} h_E \left\| \frac{\partial v}{\partial \nu_h} \right\|_{L^2(E)}^2$$

$$\|v\|_{DG}^2 := \|\nabla v\|_{L^2(\Omega_h)}^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{\varepsilon + h_E} \|v\|_{L^2(E)}^2 + \sum_{E \in \mathcal{I}_h} \frac{1}{h_E} \|[\![v]\!]\|_{L^2(E)}^2$$

$$\|v\|_{DG,h}^2 := \|v\|_{DG}^2 + \sum_{E \in \mathcal{I}_h \cup \mathcal{E}_h} h_E \|\{\!\{\nabla v\}\!\}\|_{L^2(E)}^2$$

Properties of bilinear forms

Lem. 2 (Continuity, Coercivity)

We obtain

$$a_h^{\#}(w,v) \le C \|w\|_{\#,h} \|v\|_{\#,h} \quad {}^{\forall} w,\, v \in H^s(\Omega_h) + V_{\#}.$$

Moreover, If γ is sufficiently small, we have

$$a_h^{\#}(\chi,\chi) \ge C \|\chi\|_{\#}^2 \quad {}^{\forall}\chi \in V_{\#}.$$

Generally, $u \notin H^s(\Omega_h)$. Even if $\tilde{u}$ is extension of u, $a_h(\tilde{u},v) \neq l_h(v)$

Lem. 3

Let $u \in H^2(\Omega)$, $u_\# \in V_\#$ be the solution of (1), (#), respectively. Then, if γ is sufficiently small, the following estimates hold.

$$\|\tilde{u} - u_\#\|_{\#,h} \leq C \left[\inf_{\xi \in V_\#} \|\tilde{u} - \xi\|_{\#,h} + \sup_{\chi \in V_\#} \frac{|a_h^\#(\tilde{u}, \chi) - l_h(\chi)|}{\|\chi\|_\#} \right] \tag{7}$$

$$\begin{aligned} \|\tilde{u} - u_\#\|_{L^2(\Omega_h)} \leq C \Bigg[& \|\tilde{u} - u_\#\|_{L^2(\Omega_h \setminus \Omega)} + h \|\tilde{u} - u_\#\|_{\#,h} \\ & + \sup_{z \in H^2(\Omega)} \Bigg(\frac{\|\tilde{z} - \Pi_\# \tilde{z}\|_{\#,h} \|\tilde{u} - u_\#\|_{\#,h}}{\|z\|_{H^2(\Omega)}} \\ & + \frac{|a_h^\#(\tilde{u}, \Pi_\# \tilde{z}) - l_h(\Pi_\# \tilde{z})|}{\|z\|_{H^2(\Omega)}} \Bigg) \Bigg] \end{aligned} \tag{8}$$

$\tilde{u}$, $\tilde{z}$: extension of u, z.

Proof. Step 1

For $\xi \in V_\#$, $\chi = u_\# - \xi$. By properties of bilinear form,

$$
\begin{aligned}
\|\chi\|_\#^2 &\leq C a_h^\#(\chi, \chi) = C(l_h(\chi) - a_h^\#(\xi, \chi)) \\
&= C(a_h^\#(\tilde{u} - \xi, \chi) - a_h^\#(\tilde{u}, \chi) + l_h(\chi)) \\
&\leq C \|\tilde{u} - \xi\|_{\#,h} \|\chi\|_\# + \left| a_h^\#(\tilde{u}, \chi) - l_h(\chi) \right|.
\end{aligned}
$$

Divide this by $\|\chi\|_\#$, (7) holds.

Error estimates

Proof. Step 2

For $\eta \in L^2(\Omega_h)$, $\tilde{\eta} \in L^2(\tilde{\Omega})$ denotes 0 extension. Let $z \in H^2(\Omega)$ be the solution of

$$\left\{ \begin{array}{rcccc} -\Delta z & = & \tilde{\eta} & \text{in} & \Omega \\ \frac{\partial z}{\partial \nu} + z/\varepsilon & = & 0 & \text{on} & \Gamma \end{array} \right. .$$

Then,

$$\begin{aligned} (\tilde{u} - u_\#, \eta)_{\Omega_h} = {} & a_h^\#(\tilde{u} - u_\#, \tilde{z} - \Pi_\# \tilde{z}) \\ & + a_h^\#(\tilde{u}, \Pi_\# \tilde{z}) - l_h(\Pi_\# \tilde{z}) + (\tilde{u} - u_\#, \eta + \Delta \tilde{z})_{\Omega_h \setminus \Omega} \\ & - \sum_{E \in E_h} \left[\frac{\varepsilon}{\varepsilon + \gamma h_E} \langle \frac{\partial \tilde{z}}{\partial \nu_h} + \frac{\tilde{z}}{\varepsilon}, (1 - \gamma h_E \frac{\partial}{\partial \nu_h})(\tilde{u} - u_\#) \rangle_E \right]. \end{aligned}$$

By boundary-skin estimates and other estimates, we have (8).

Thm. 1 (Optimal energy error)

Let $u \in H^3(\Omega)$ be the solution of (1). Let $u_\# \in V_\#$ be the solution of $(\#)$. Then, we have

$$\|\tilde{u} - u_\#\|_{\#,h} \le Ch(\|u\|_{H^3(\Omega)} + \|\tilde{u}_0\|_{H^1(\tilde{\Omega})} + \|\tilde{g}\|_{H^1(\tilde{\Omega})}). \tag{9}$$

Proof. Step 1

Need to estimate $\left|a_h^\#(\tilde{u}, \chi) - l_h(\chi)\right|$.

$$\begin{aligned}\left|a_h^\#(\tilde{u}, \chi) - l_h(\chi)\right| \le{}& |(-\Delta\tilde{u} - \tilde{f}, \chi)_{\Omega_h}| \\ &+ \sum_{E \in \mathcal{E}_h} \frac{\varepsilon}{\varepsilon + \gamma h_E} \left|\langle \frac{\partial \tilde{u}}{\partial \nu_h} + \frac{\tilde{u} - \tilde{u}_0}{\varepsilon} - \tilde{g}, \chi - \gamma h_E \frac{\partial \chi}{\partial \nu_h}\rangle_E\right|\end{aligned}$$

Error estimates

Proof. Step 2

By boundary-skin estimates, $|(-\Delta \tilde{u} - \tilde{f}, \chi)_{\Omega_h}| \leq Ch^2 \|u\|_{H^3(\Omega)} \|\chi\|_{\#,h}$.
By Hölder's inequality and boundary-skin estimates,

$$\begin{aligned}
&\sum_{E \in \mathcal{E}_h} \frac{\varepsilon}{\varepsilon + \gamma h_E} \left| \langle \frac{\partial \tilde{u}}{\partial \nu_h} + \frac{\tilde{u} - \tilde{u}_0}{\varepsilon} - \tilde{g}, \chi - \gamma h_E \frac{\partial \chi}{\partial \nu_h} \rangle_E \right| \\
&\leq C \left\| \frac{\partial \tilde{u}}{\partial \nu_h} + \frac{\tilde{u} - \tilde{u}_0}{\varepsilon} - \tilde{g} \right\|_{L^2(\Gamma_h)} \|\chi\|_{\#,h} \\
&\leq Ch(\|u\|_{H^2(\Omega)} + \|\tilde{u}_0\|_{H^1(\tilde{\Omega})} + \|\tilde{g}\|_{H^1(\tilde{\Omega})}) \|\chi\|_{\#,h} .
\end{aligned}$$

$\|\chi\|_{\#}$ and $\|\chi\|_{\#,h}$ are uniformly equivalent on $V_{\#}$, we obtain (9).

Error estimates

Cor. 1 (Spherically symmetric case)

Let $\Omega = \{x \in \mathbb{R}^d \colon \ |x| < 1\}$.
Assume that the solution $u \in H^3(\Omega)$ of (1) is spherically symmetric.
Define two finite elements spaces

$$V_{N,2} := \{\chi \in C(\overline{\Omega}) \colon \chi|_K \in \mathcal{P}^2(K) \, {}^\forall K \in \mathcal{T}_h\}$$

$$V_{DG,2} := \{\chi \in L^2(\Omega) \colon \chi|_K \in \mathcal{P}^2(K) \, {}^\forall K \in \mathcal{T}_h\}.$$

Let $u_\# \in V_{\#,2}$ be the solution of

$$a_h^\#(u_\#, \chi) = l_h(\chi) \quad {}^\forall \chi \in V_{\#,2}.$$

Then,

$$\|\tilde{u} - u_\#\|_{\#,h} \le Ch^2(\|u\|_{H^3(\Omega)} + \|\tilde{u}_0\|_{H^2(\widetilde{\Omega})} + \|\tilde{g}\|_{H^2(\widetilde{\Omega})})$$

Thm. 2 (Optimal L^2 error)

Let $u \in H^4(\Omega)$, $u_\# \in V_\#$ be the solution of (1), (#), respectively. Then, we have

$$\|\tilde{u} - u_\#\|_{L^2(\Omega_h)} \leq Ch^2(\|u\|_{H^4(\Omega)} + \|\tilde{u}_0\|_{H^3(\widetilde{\Omega})} + \|\tilde{g}\|_{H^3(\widetilde{\Omega})}). \tag{10}$$

Proof. Step 1

$$b(w,v) = \sum_{E \in \mathcal{E}_h} \Big\{ -\frac{\gamma h_E}{\varepsilon + \gamma h_E} \Big(\langle \frac{\partial w}{\partial \nu}, v \rangle_{\pi(E)} + \langle w, \frac{\partial v}{\partial \nu} \rangle_{\pi(E)} \Big) + \frac{1}{\varepsilon + \gamma h_E} \langle w, v \rangle_{\pi(E)} - \frac{\varepsilon \gamma h_E}{\varepsilon + \gamma h_E} \langle \frac{\partial w}{\partial \nu}, \frac{\partial v}{\partial \nu} \rangle_{\pi(E)} \Big\}$$

$$l(v) = (f, v)_\Omega + \sum_{E \in \mathcal{E}_h} \Big\{ \frac{1}{\varepsilon + \gamma h_E} \langle u_0 + \varepsilon g, v - \gamma h_E \frac{\partial v}{\partial \nu} \rangle_{\pi(E)} \Big\}$$

Error estimates

Proof. Step 2

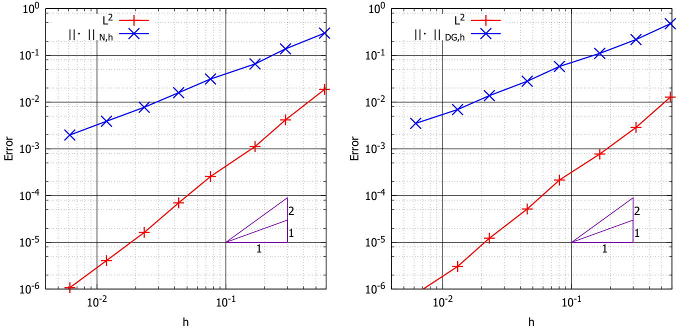
Define $a(w, v) = (\nabla w, \nabla v)_\Omega + b(w, v)$.
Then, for $v \in H^s(\Omega) + V_\#$, $a(u, v) = l(v)$ holds.
Since $\tilde{u}$, $\tilde{z}$ are continuous on Ω_h, $J_h(\tilde{u}, \tilde{z}) = 0$. Hence,

$$\begin{aligned} a_h^\#(\tilde{u}, \Pi_\# \tilde{z}) - l_h(\Pi_\# \tilde{z}) &= a_h^\#(\tilde{u}, \Pi_\# \tilde{z} - \tilde{z}) - l_h(\Pi_\# \tilde{z} - \tilde{z}) \\ &\quad + a_h^\#(\tilde{u}, \tilde{z}) - a(u, z) + l(z) - l_h(\tilde{z}). \end{aligned}$$

For first line, we get $O(h^2)$ estimate by consistency error and interpolation estimate.
Second line includes terms such as $\int_{\Omega_h \setminus \Omega} \tilde{\cdot}\, dx$, $\int_{\Omega \setminus \Omega_h} \cdot\, dx$,
$\int_E \tilde{\cdot}\, d\gamma - \int_{\pi(E)} \cdot\, d\gamma$.
We get estimate (10) by boundary-skin estimates.

4 . Numerical examples

$\Omega = \{|x| < 1\} \subset \mathbb{R}^2,\ \ u(x) = \sin(x_1)\sin(x_2)$



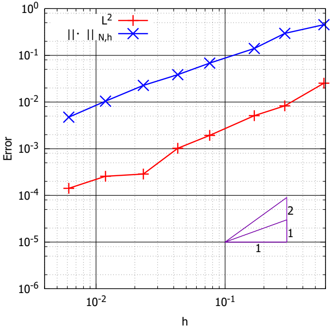
(a) Nitsche (b) DG

Errors of energy and L^2 norms

Energy norm: $O(h)$, L^2 norm: $O(h^2)$

Numerical examples

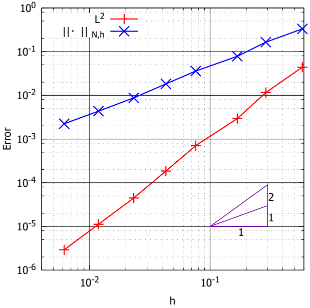
$\Omega = \{|x| < 1\} \subset \mathbb{R}^2$, $u(x) = \sqrt{(x_1+1)^2 + x_2^2} \quad (u \notin H^4)$



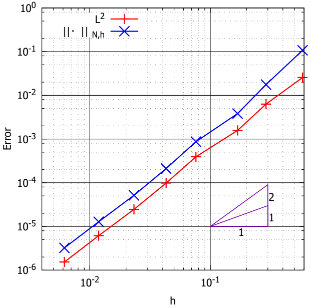
Errors of energy and L^2 norms of Nitsche's method

Energy norm and L^2 norm: $O(h)$

$\Omega = \{|x| < 1\} \subset \mathbb{R}^2$, $u(x) = \exp(-x_1^2 - x_2^2)$ (spherically symmetric)



(a) P1 Element



(b) P2 Element

Errors of energy and L^2 norms of Nitsche's method

P1 Element: Energy norm: $O(h)$, L^2 norm $O(h^2)$

P2 Element: Energy norm and L^2 norm: $O(h^2)$

Conclusion

We show the energy and L^2 error estimates of Nitsche's and DG methods for the Poisson equation with Robin boundary condition in a smooth domain.

The results are optimal for the P1 elements, and the energy error is optimal for the P2 elements in the case of a spherically symmetric function.

Future works

- Extend these results to generalized Robin and dynamic boundary conditions