Hybridized discontinuous Galerkin method for a PDE in a smooth domain

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RIMS 共同研究 諸科学分野を結ぶ基礎学問としての数値解析学
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1 . Introduction

HDG method

Finite element method

Continuous piecewise polynomial

Discontinous Galerkin method

Discontinuous piecewise polynomial
Good for convection dominant problem. DoF is relatively large.

Hybridized discontinuous Galerkin(HDG) method

Discontinuous piecewise polynomial and numerical trace between each element
DoF is smaller than that of DG using higher degree polynomial(static condensation).
Superconvergence properties hold under some conditions.

HDG method

Static condensation

u_h: polynomial on each element, $\hat{u}_h$: numerical trace

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u_h \\ \hat{u}_h \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

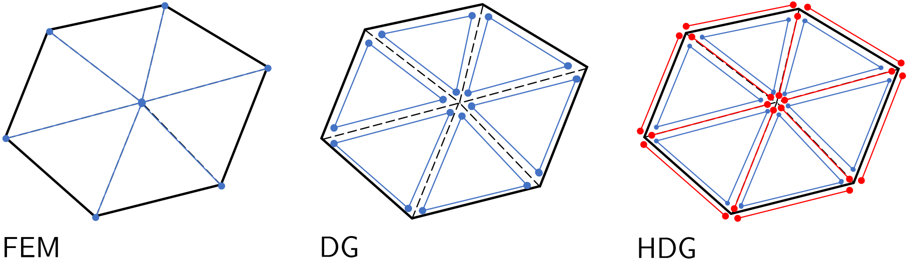
$$\Leftrightarrow A_{22}\hat{u}_h = G - A_{21}u_h,\ A_{11}u_h = F - A_{12}\hat{u}_h$$

$$\Leftrightarrow \underbrace{(A_{22} + A_{21}A_{11}^{-1}A_{12})\hat{u}_h = G - A_{21}A_{11}^{-1}F}_{\text{Global problem}},\ \underbrace{u_h = A_{11}^{-1}(F - A_{12}\hat{u}_h)}_{\text{Local problem}}$$

Local problem can be solved independently on each element.

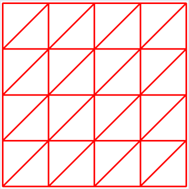
$\rightarrow$The cost of calculation A_{11}^{-1} is small.

HDG method



Degree of freedom (2-dimension space)

	FEM	DG	HDG
P1	vertex (25)	$3\times$element (96)	$2\times$edge (112)
P2	vertex$+$edge (81)	$6\times$element (192)	$3\times$edge (168)
P3	vertex$+2\times$edge $+$element (169)	$10\times$element(320)	$4\times$edge(224)



Numerical method in a smooth domain

Numerical computitaion in a smooth domain

We utilize polyhedral approximations of the domain.

Original problem

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad {}^\forall v \in V$$

→

Approximate problem

$$\int_{\Omega_h} \nabla u \cdot \nabla v \, dx = \int_{\Omega_h} \tilde{f} v \, dx \quad {}^\forall v \in V_h$$

Even if Ω_h converges to Ω, an approximate solution may converge to the solution of a different problem.

Numerical method in a smooth domain

Example 1. Babsŭka's paradox

$$\begin{cases} -\Delta^2 u & = & 1 & \text{in} & \Omega \\ u = \frac{\partial^2 u}{\partial \nu^2} & = & 0 & \text{on} & \partial\Omega \end{cases} \qquad \text{(A)}$$

$\Omega = \{x^2 + y^2 < 1\}$,$\nu$: outward unit normal vector

Exact solution $u(x, y) = (x^2 + y^2 - 1)(x^2 + y^2 - 5)/64$

$$\begin{cases} -\Delta^2 u_n & = & 1 & \text{in} & \Omega_n \\ u_n = \frac{\partial^2 u_n}{\partial \nu_n^2} & = & 0 & \text{on} & \partial\Omega_n \end{cases} \qquad \text{(B}_n\text{)}$$

Ω_n: regular n-polygon inscribed to Ω,ν_n: outward unit normal vector

$u_n(x, y) \to (x^2 + y^2 - 1)(x^2 + y^2 - 3)/64 \neq u(x, y)$ as $n \to \infty$

Example 2. generalized Robin boundary conditon

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} + u - \Delta_\Gamma u = g & \text{on} \quad \Gamma \end{cases} \tag{C}$$

Ω: smooth domain, Δ_Γ: Laplace-Beltrami operator

Apply two DG schemes

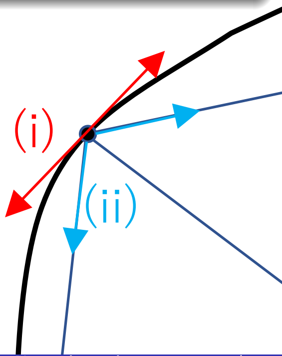
Difference $\rightarrow$ definition of jump of boundary

(i) Use Ω

(ii) Use Ω_h

Scheme (i): $u_h \rightarrow u$

Scheme (ii): $u_h \not\rightarrow u$



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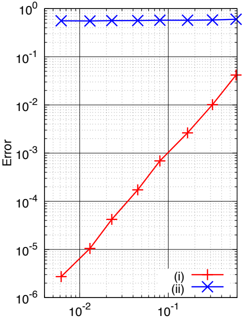

eg. Ω: circle

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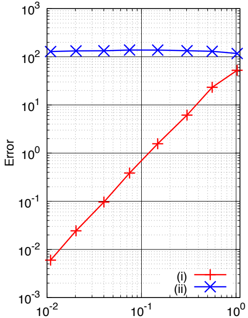


eg. Ω: ellipse

2 . HDG Scheme

Model equation

Diffusion equation

$$\begin{cases} -\nabla \cdot (c^{-1}\nabla u) &= f \quad \text{in} \quad \Omega \\ c^{-1}\dfrac{\partial u}{\partial \nu} + \alpha u &= g \quad \text{on} \quad \Gamma = \partial\Omega \end{cases} \tag{1}$$

$\Omega \subset \mathbb{R}^N (N = 2, 3)$: smooth domain, ν: outward unit normal vector on Γ
c: sufficiently smooth, symmetric, uniformly positive definite matrix-valued function
f, g: sufficiently smooth function, α: positive constant

This is equivalent to

$$\begin{cases} cq + \nabla u &= 0 \quad \text{in} \quad \Omega \\ \nabla \cdot q &= f \quad \text{in} \quad \Omega \\ -q \cdot \nu + \alpha u &= g \quad \text{on} \quad \Gamma \end{cases} \tag{2}$$

Polygonal approximation

$\{\mathcal{T}_h\}_h$: a family of regular triangulations, $\partial\mathcal{T}_h := \{\partial K : K \in \mathcal{T}_h\}$
$\Omega_h = \mathrm{int}(\bigcup_{K\in\mathcal{T}_h} \overline{K})$, $\Gamma_h = \partial\Omega_h$.

$$\mathcal{F}_h := \{F : F \text{ is an } (N-1)\text{-face of some } K \in \mathcal{T}_h\}$$
$$\mathcal{F}_h^\partial := \{F \in \mathcal{F}_h : F \subset \Gamma_h\}, \quad \mathcal{F}_h^i := \{F \in \mathcal{F}_h : F \not\subset \Gamma_h\} = \mathcal{F}_h \backslash \mathcal{F}_h^\partial.$$

Assume that

Every vertex of $F \in \mathcal{F}_h^\partial$ lies on Γ.

Γ_h is expressed as $\Gamma_h = \bigcup_{F\in\mathcal{F}_h^\partial} F$.
For the sake of simplicity,

$$(w,v)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (w,v)_K, \quad \langle w,v\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \langle w,v\rangle_{\partial K}$$
$$\|w\|^2_{\mathcal{T}_h} = (w,w)_{\mathcal{T}_h}, \quad \|w\|^2_{\partial\mathcal{T}_h} = \langle w,w\rangle_{\partial\mathcal{T}_h}.$$

HDG spaces

HDG spaces

$$W_h := \{w_h \in L^2(\Omega_h) \colon w_h|_K \in W_h(K)\ {}^\forall K \in \mathcal{T}_h\}$$
$$V_h := \{v_h \in L^2(\Omega_h)^N \colon v_h|_K \in V_h(K)\ {}^\forall K \in \mathcal{T}_h\}$$
$$M_h := \{\mu_h \in L^2(\mathcal{F}_h) \colon \mu_h|_K \in M_h(F)\ {}^\forall F \in \mathcal{F}_h\}$$

Here,

$$W_h(K) := \{w_h \in L^2(K) \colon w_h \in \mathcal{P}^1(K)\}$$
$$V_h(K) := \{v_h \in L^2(K)^N \colon v_h \in \mathcal{P}^1(K)^N\}$$
$$M_h(F) := \{\mu_h \in L^2(F) \colon \mu_h \in \mathcal{P}^1(F)\}$$
$$M_h(\partial K) := \{\mu_h \in L^2(\partial K) \colon \mu_h|_F \in M_h(F)\ {}^\forall F \in \mathcal{F}_h,\ F \subset \partial K\}$$

for $K \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$.

HDG scheme

For $q_h \in V_h(K)$, $u_h \in W_h(K)$ and $\hat{u}_h \in M_h(\partial K)$,

$$\hat{q}_h \cdot n := q_h \cdot n + \tau(u_h - \hat{u}_h) \quad \text{on } \partial K$$

n: outward unit normal vector on ∂K,
τ: symmetric uniformly positive linear function.

HDG scheme

Find $(q_h, u_h, \hat{u}_h) \in V_h \times W_h \times M_h$ s.t.

$$\begin{cases} (cq_h, v_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot v_h)_{\mathcal{T}_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial \mathcal{T}_h} = 0 & {}^\forall v_h \in V_h \\ -(q_h, \nabla w_h)_{\mathcal{T}_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial \mathcal{T}_h} = (\tilde{f}, w)_{\Omega_h} & {}^\forall w_h \in W_h \\ -\langle \hat{q}_h \cdot n, \mu_h \rangle_{\partial \mathcal{T}_h} + \alpha \langle \hat{u}_h, \mu_h \rangle_{\Gamma_h} = \langle \tilde{g}, \mu_h \rangle_{\Gamma_h} & {}^\forall \mu_h \in M_h \end{cases} \tag{3}$$

$\tilde{f}$, $\tilde{g}$: extension of f, g

HDG scheme

P_V is projection into V_h satisfying

$$(cP_V q, v_h)_{\mathcal{T}_h} = (cq, v_h)_{\mathcal{T}_h} \quad {}^\forall v_h \in V_h$$

For $\mu \in L^2(\partial\mathcal{T}_h)$, we define lift function $\Phi(\mu) \in V_h$ as

$$(c\Phi(\mu), v_h)_{\mathcal{T}_h} = \langle \mu, v_h \cdot n \rangle_{\partial\mathcal{T}_h} \quad {}^\forall v_h \in V_h.$$

Bilinear form a_h in $(W_h + H^s(\Omega_h)) \times L^2(\mathcal{F}_h)$ as

$$\begin{aligned} a_h(u, \hat{u}; w, \mu) &= (cP_V c^{-1}\nabla u, P_V c^{-1}\nabla w)_{\mathcal{T}_h} - \langle (P_V c^{-1}\nabla u)\cdot n, w - \mu\rangle_{\partial\mathcal{T}_h} \\ &\quad - \langle u - \hat{u}, (P_V c^{-1}\nabla w)\cdot n\rangle_{\partial\mathcal{T}_h} + (c\Phi(u-\hat{u}), \Phi(w-\mu))_{\mathcal{T}_h} \\ &\quad + \langle \tau(u-\hat{u}), w-\mu\rangle_{\partial\mathcal{T}_h} + \alpha\langle \hat{u}, \mu\rangle_{\Gamma_h} \\ l_h(w_h, \mu_h) &= (\tilde{f}, w_h)_{\Omega_h} + \alpha\langle \tilde{g}, \mu_h\rangle_{\Gamma_h} \end{aligned}$$

Scheme (3) can be rewritten as

$$\text{Find} \quad (u_h, \hat{u}_h) \in W_h \times M_h \quad \text{s.t.}$$
$$a_h(u_h, \hat{u}_h; w_h, \mu_h) = l_h(w_h, \mu_h) \quad {}^\forall w_h \in W_h, \mu \in M_h \tag{4}$$

3 . Analysis of the scheme

Lemma 1 (Boundary-skin estimates(Kashiwabara-Oikawa-Zhou(2016), Kashiwabara-Kemmochi(preprint)))

For sufficiently small h, the following estimates hold.

$$\left| \int_{\pi(F)} f \, d\gamma - \int_F f \circ \pi \, d\gamma_h \right| \leq Ch^2 \int_{\pi(F)} |f| \, d\gamma \quad f \in L^1(\pi(F)), \ F \in \mathcal{F}_h^\partial \tag{5}$$

$$\|f - f \circ \pi\|_{L^p(\Gamma_h)} \leq Ch^{2-2/p} \|f\|_{W^{1,p}(\Gamma(Ch^2))} \quad f \in W^{1,p}(\Gamma(Ch^2)) \tag{6}$$

$$\|f\|_{L^p(\Gamma(Ch^2))} \leq C(h^2 \|\nabla f\|_{L^p(\Gamma(Ch^2))} + h^{2/p} \|f\|_{L^p(\Gamma)}) \quad f \in W^{1,p}(\Gamma(Ch^2)) \tag{7}$$

$$\|f\|_{L^p(\Omega_h \setminus \Omega)} \leq C(h^2 \|\nabla f\|_{L^p(\Omega_h \setminus \Omega)} + h^{2/p} \|f\|_{L^p(\Gamma_h)}) \quad f \in W^{1,p}(\Omega_h) \tag{8}$$

$$\|\nu_h - \nu \circ \pi\|_{L^\infty(\Gamma_h)} \leq Ch \tag{9}$$

π: orthogonal projection into Γ,ν_h: outward unit normal vector of Γ_h
$\Gamma(Ch^2)$: neighborhood of Γ

Boundary-skin estimates

Lemma 2

For $w \in H^1(\Omega_h)$, $w_h \in W_h$ and $\mu_h \in M_h$

$$\|w+w_h\|_{L^2(\Omega_h\setminus\Omega)} \leq Ch\|w+w_h\|_{H^1(\Omega_h)}+h\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|w_h-\mu_h\|^2_{L^2(\partial K)}\right)^{1/2} \tag{10}$$

Define $E_h\mu_h \in W_H \cap H^1(\Omega_h)$ as

$$E_h\mu_h(p) = \frac{1}{\mathcal{F}_h(p)} \sum_{F\in\mathcal{F}_h(p)} \mu_h|_F(p) \quad p \text{ is a vertex of } K \in \mathcal{T}_h$$

where $\mathcal{F}_h(p) = \{F \in \mathcal{F}_h : p \in \partial F\}$.

$$\begin{aligned}\|w + w_h\|_{L^2(\Omega_h\setminus\Omega)} \leq Ch\|w + w_h\|_{H^1(\Omega_h)} + Ch\|w_h - E_h\mu_h\|_{H^1(\Omega_h)} \\ + \|w_h - E_h\mu_h\|_{L^2(\Omega_h\setminus\Omega)}\end{aligned}$$

Properties of HDG space

Remark 1

For $w \in H^2(\Omega_h)$, there exists $(w_h, \mu_h) \in W_h \times M_h$ such that

$$\|w - w_h\|_{H^k(K)} \leq Ch^{2-k}\|w\|_{H^2(K)} \quad {}^\forall K \in \mathcal{T}_h,\ k = 0, 1, 2$$
$$\|w - \mu_h\|_{L^2(\partial K)} \leq Ch_K^{1+1/2}\|w\|_{H^2(K)} \quad {}^\forall K \in \mathcal{T}_h$$

Remark 2

There exists a positive constant C_1 satisfying

$$\|v_h\|^2_{L^2(\partial K)} \leq C_1 h_K^{-1}\|v_h\|^2_{L^2(K)} \quad {}^\forall v_h \in V_h(K), K \in \mathcal{T}_h. \tag{11}$$

HDG norm

$$\|\{w, \mu\}\|_h^2 := \|c^{1/2}P_V c^{-1}\nabla w\|^2_{\mathcal{T}_h} + \|\tau^{1/2}(w - \mu)\|^2_{\partial\mathcal{T}_h} + \alpha\|\mu\|^2_{\Gamma_h}$$

Properties of a_h

For $K \in \mathcal{T}_h$,

$$c_{K,\min} := \min_{w \in L^2(K)^2 \setminus \{0\}} \frac{(cw, w)_K}{\|c\|^2_{L^2(K)}}, \quad \tau_{K,\min} := \min_{\mu \in M_h(K) \setminus \{0\}} \frac{\langle \tau\mu, \mu \rangle_{\partial K}}{\|\mu\|^2_{L^2(\partial K)}}$$

Assume that there exists a constant $M > 1$ satisfying

$$\tau_{K,\min} c_{K,\min} > M C_1 h_K^{-1} \quad {}^\forall K \in \mathcal{T}_h, \tag{12}$$

Lemma 3

There exists a positive constant C satisfying

$$a_h(u, \hat{u}; w, \mu) \le C \|\{u, \hat{u}\}\|_h \|\{w, \mu\}\|_h$$
$$ {}^\forall (u, \hat{u}), (w, \mu) \in (W_h + H^s(\Omega_h)) \times L^2(\mathcal{F}_h), \tag{13}$$

and

$$a_h(w_h, \mu_h; w_h, \mu_h) \ge C \|\{w_h, \mu_h\}\|_h^2 \quad {}^\forall (w_h, \mu_h) \in W_h \times M_h. \tag{14}$$

Energy error estimate

Theorem 1

Let $u \in H^3(\Omega)$ be the solution of (1), and $(u_h, \hat{u}_h) \in W_h \times M_h$ be the solution of (4). Then,

$$\|\{\tilde{u} - u_h, \tilde{u} - \hat{u}_h\}\|_h \leq Ch(\|u\|_{H^3(\Omega)} + \|\tilde{g}\|_{H^1(\widetilde{\Omega})}). \tag{15}$$

Here, $\tilde{u}$ is extension of u.

Using (13) and (14), we have

$$\begin{aligned}\|\{\tilde{u} - u_h, \tilde{u} - \hat{u}_h\}\|_h \leq C &\inf_{(w_h,\mu_h)\in W_h\times M_h} \|\{\tilde{u} - w_h, \tilde{u} - \mu_h\}\|_h \\ &+ C \sup_{(w_h,\mu_h)\in W_h\times M_h} \frac{|a_h(\tilde{u}, \tilde{u}; w_h, \mu_h) - l_h(w_h, \mu_h)|}{\|\{w_h, \mu_h\}\|_h}\end{aligned}$$

Energy error estiamte

Using integration by parts, we obtain

$$
\begin{aligned}
a_h(\tilde{u}, \tilde{u}; w_h, \mu_h) &- l_h(w_h, \mu_h) \\
&= (\nabla \cdot \theta, w_h)_{\mathcal{T}_h} - \langle \theta, \mu_h \rangle_{\mathcal{T}_h} + (-\nabla \cdot c^{-1} \nabla \tilde{u} - \tilde{f}, w_h)_{\mathcal{T}_h} \\
&\quad + \langle c^{-1} \nabla \tilde{u} \cdot n + \alpha \tilde{u} - \tilde{g}, \mu_h \rangle_{\Gamma_h}
\end{aligned}
$$

where $\theta = c^{-1} \nabla \tilde{u} - P_V c^{-1} \nabla \tilde{u}$.

$$
\begin{aligned}
&|(\nabla \cdot \theta, w_h)_{\mathcal{T}_h} - \langle \theta, \mu_h \rangle_{\mathcal{T}_h}| \leq |-(\theta, \nabla w_h)_{\mathcal{T}_h} + \langle \theta, w_h - \mu_h \rangle_{\mathcal{T}_h}| \\
&\leq Ch \|\tilde{u}\|_{H^2(\Omega_h)} \|\nabla w_h\|_{\mathcal{T}_h} + (\sum_{K \in T_h} \|\theta\|^2_{L^2(\partial K)})^{1/2} (\sum_{K \in T_h} \|w_h - \mu_h\|^2_{L^2(\partial K)})^{1/2} \\
&\leq Ch \|\tilde{u}\|_{H^2(\Omega_h)} (\|c^{1/2} P_V c^{-1} \nabla w_h\|_{\mathcal{T}_h} + \|\tau^{1/2}(w_h - \mu_h)\|_{\partial \mathcal{T}_h}).
\end{aligned}
$$

L^2 error estimate

Theorem 2

Let $u \in H^4(\Omega)$ be the solution of (1), and $(u_h, \hat{u}_h) \in W_h \times M_h$ be the solution of (4). Then,

$$\|\tilde{u} - u_h\|_{L^2(\Omega_h)} \leq Ch^2(\|u\|_{H^4(\Omega)} + \|\tilde{g}\|_{H^3(\tilde{\Omega})}), \tag{16}$$

For $\eta \in L^2(\Omega_h)$, we define $\tilde{\eta}$ as zero extension. Let $z \in H^2(\Omega)$ be the solution of

$$\begin{cases} -\nabla \cdot (c^{-1}\nabla z) &= \tilde{\eta} \quad \text{in} \quad \Omega \\ c^{-1}\dfrac{\partial z}{\partial \nu} + \alpha z &= 0 \quad \text{on} \quad \Gamma. \end{cases}$$

Then,

$$\|z\|_{H^2(\Omega)} \leq C\|\eta\|_{L^2(\Omega)}. \tag{17}$$

L^2 error estimate

We set $\tilde{z}$ is extension of z, $e = \tilde{u} - u_h$ and $\hat{e} = \tilde{u} - \hat{u}_h$. Then, we have

$$
\begin{aligned}
(e, \eta)_{L^2(\Omega_h)} &= (e, -\nabla \cdot (c^{-1} \nabla \tilde{z}))_{\Omega_h} - (e, -\nabla \cdot (c^{-1} \nabla \tilde{z}))_{\Omega_h} + (e, \eta)_{L^2(\Omega_h)} \\
&= (e, -\nabla \cdot (c^{-1} \nabla \tilde{z}))_{\Omega_h} + (e, \eta + \nabla \cdot (c^{-1} \nabla \tilde{z}))_{L^2(\Omega_h \setminus \Omega)}.
\end{aligned}
$$

Using integration by parts, we obtain

$$
\begin{aligned}
&(e, -\nabla \cdot (c^{-1} \nabla \tilde{z}))_{\Omega_h} \\
&= (\nabla e, c^{-1} \nabla \tilde{z})_{\mathcal{T}_h} - \langle e, c^{-1} \nabla \tilde{z} \cdot n \rangle_{\partial \mathcal{T}_h} \\
&= a_h(e, \hat{e}; \tilde{z} - w_h, \tilde{z} - \mu_h) + a_h(\tilde{u}, \tilde{u}; w_h, \mu_h) - l_h(w_h, \mu_h) \\
&\qquad + (\nabla e, c^{-1} \nabla \tilde{z} - P_V c^{-1} \nabla \tilde{z})_{\mathcal{T}_h} + \langle e - \hat{e}, P_V c^{-1} \nabla \tilde{z} \cdot n - c^{-1} \nabla \tilde{z} \cdot n \rangle_{\partial \mathcal{T}_h} \\
&\qquad - \langle \hat{e}, c^{-1} \nabla z \cdot \nu + \alpha \tilde{z} \rangle_{L^2(\Gamma_h)}
\end{aligned}
$$

where $w_h \in W_h$ and $\mu_h \in M_h$.

L^2 estimate

$$a_h(\tilde{u}, \tilde{u}; w_h, \mu_h) - l_h(w_h, \mu_h) = a_h(\tilde{u}, \tilde{u}; \tilde{z}, \tilde{z}) - l_h(\tilde{z}, \tilde{z}) \\ + a_h(\tilde{u}, \tilde{u}; \tilde{z} - w_h, \tilde{z} - \mu_h) - l_h(\tilde{z} - w_h, \tilde{z} - \mu_h)$$

Since u is the solution of (1) and $\tilde{u} = Pu$, we have

$$(c^{-1}\nabla\tilde{u}, \nabla\tilde{z})_\Omega + \alpha\langle\tilde{u}, \tilde{z}\rangle_\Gamma = (\tilde{f}, \tilde{z})_\Omega + \langle\tilde{g}, \tilde{z}\rangle_\Gamma.$$

Hence, we obtain

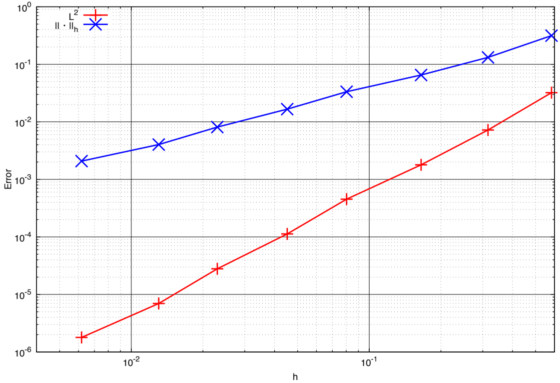
$$a_h(\tilde{u}, \tilde{u}; \tilde{z}, \tilde{z}) - l_h(\tilde{z}, \tilde{z}) = (cP_V c^{-1}\nabla\tilde{u}, P_V c^{-1}\nabla\tilde{z})_{\mathcal{T}_h} - (c^{-1}\nabla\tilde{u}, \nabla\tilde{z})_\Omega \\ + \alpha(\langle\tilde{u}, \tilde{z}\rangle_{\Gamma_h} - \langle\tilde{u}, \tilde{z}\rangle_\Gamma) - ((\tilde{f}, \tilde{z})_{\Omega_h} - (\tilde{f}, \tilde{z})_\Omega) - \alpha(\langle\tilde{g}, \tilde{z}\rangle_{\Gamma_h} - \langle\tilde{g}, \tilde{z}\rangle_\Gamma),$$

and we have

$$(cP_V c^{-1}\nabla\tilde{u}, P_V c^{-1}\nabla\tilde{z})_{\mathcal{T}_h} - (c^{-1}\nabla\tilde{u}, \nabla\tilde{z})_\Omega \\ = (P_V c^{-1}\nabla\tilde{u} - c^{-1}\nabla\tilde{u}, \nabla\tilde{z})_{\mathcal{T}_h} + (c^{-1}\nabla\tilde{u}, \nabla\tilde{z})_{\Omega_h} - (c^{-1}\nabla\tilde{u}, \nabla\tilde{z})_\Omega$$

4 . Numerical examples

$\Omega = \{|x| < 1\} \subset \mathbb{R}^2$, exact solution: $u(x) = \exp(-|x|^2)$

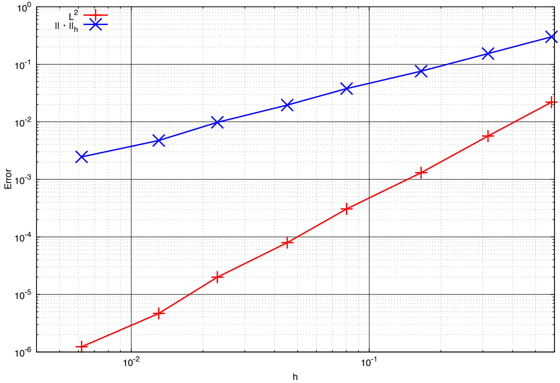


Energy and L^2 norm error

Energy norm: $O(h)$, L^2 norm: $O(h^2)$

Numerical examples

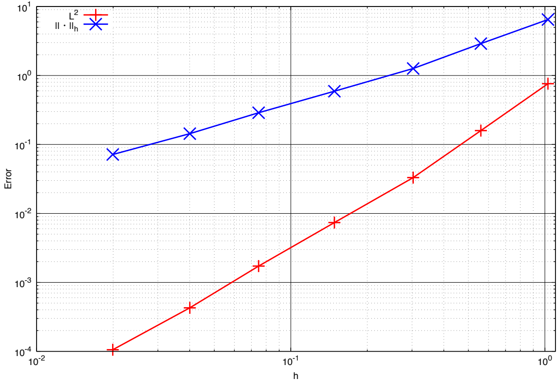
$\Omega = \{|x| < 1\} \subset \mathbb{R}^2$, exact solution: $u(x) = \sin(x_1)\sin(x_2)$



Energy and L^2 norm error

Energy norm: $O(h)$, L^2 norm: $O(h^2)$

$\Omega = \{x_1^2/4 + x_2^2 < 1\} \subset \mathbb{R}^2$, exact solution: $u(x) = \sin(\pi x_1)\sin(\pi x_2)$



Energy and L^2 norm error

Energy norm: $O(h)$, L^2 norm: $O(h^2)$

Conclusion

We show the energy and L^2 error estimates of HDG methods for the diffusion equation with Robin boundary condition in a smooth domain. This result is similar to previous FEM and DG results.

Future works

- Consider more complex boundary condition.
  eg. dynamic boundary condition

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \nu} + \alpha u - \beta \Delta_\Gamma u = g \quad \text{on } \Gamma$$

Appendix 1. DG methof for generalized Robin

$$\text{Find} \quad u_h \in W_h \quad \text{s.t.} \quad a_{\Omega_h}(u_h, \chi) = l_h(\chi) \quad {}^\forall \chi \in W_h \qquad (18)$$

$$a_{\Omega_h}(u,v) = a_{\mathcal{T}_h}(u,v) + a_{\mathcal{F}_h^\partial}(u,v)$$

$$a_{\mathcal{T}_h}(u,v) = \sum_{K\in\mathcal{T}_h} (\nabla u, \nabla v)_K + \sum_{F\in\mathcal{F}_h^i} \Big(-(\{\!\{\nabla u\}\!\}_F, [\![v]\!]_F)_F - ([\![v]\!]_F, \{\!\{\nabla v\}\!\}_F)_F + \frac{\sigma}{h}([\![u]\!]_F, [\![v]\!]_F)_F \Big)$$

$$a_{\mathcal{F}_h^\partial}(u,v) = \sum_{F\in\mathcal{F}_h^\partial} \Big(\alpha(u,v)_E + \beta(\nabla_{\Gamma_h} u, \nabla_{\Gamma_h} v)_F \Big) + \sum_{R\in\mathcal{R}_h} \beta\Big(-(\{\!\{\nabla_{\Gamma_h} u\}\!\}_R, [\![v]\!]_R)_R - ([\![v]\!]_R, \{\!\{\nabla_{\Gamma_h} v\}\!\}_R)_R + \frac{\sigma}{h}([\![u]\!]_R, [\![v]\!]_R)_R \Big)$$

$$l_h(v) = (\widetilde{I}_h f, v)_{\Omega_h} + (\widetilde{I}_h g, v)_{\Gamma_h}$$

$\sigma > 0$: positive constant, $\widetilde{I}_h$: Laglange interpolation into W_h.
$\mathcal{R}_h = \{R\colon R \text{ is } (n-2) - \text{face of some} F \in \mathcal{F}_h^\partial\}$

For $F \in \mathcal{F}_h^\partial$,

$$\{\!\{v\}\!\}_F := \frac{1}{2}(v_1 + v_2), \qquad [\![v]\!]_F := v_1 n_{F,1} + v_2 n_{F,2},$$

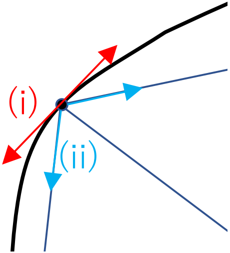
$$\{\!\{\nabla v\}\!\}_F := \frac{1}{2}(\nabla v_1 + \nabla v_2), \qquad [\![\nabla v]\!]_F := \nabla v_1 \cdot n_{F,1} + \nabla v_2 \cdot n_{F,2}.$$

Here, there exist distinct $K_1,\, K_2 \in \mathcal{T}_h$ satisfying $F = \partial K_1 \cap \partial K_2$, $v_i = v|_{K_i}$, and $n_{E,i}$ is outward unit normal vector on E with respect to K_i.
Similarly, for $R \in \mathcal{R}_h$,

$$\{\!\{v\}\!\}_R := \frac{1}{2}(v_1 + v_2), \qquad [\![v]\!]_R := v_1 n_{R,1} + v_2 n_{R,2},$$

$$\{\!\{\nabla_{\Gamma_h} v\}\!\}_R := \frac{1}{2}(\nabla_{\Gamma_h} v_1 + \nabla_{\Gamma_h} v_2), \qquad [\![\nabla_{\Gamma_h} v]\!]_R := \nabla_{\Gamma_h} v_1 \cdot n_{R,1} + \nabla_{\Gamma_h} v_2 \cdot n_{R,2}.$$

Here, there exist distinct $F_1,\, F_2 \in \mathcal{F}_h^\partial$ satsfying $R = F_1 \cap F_2$, $v_i = v|_{F_i}$.
(i) $n_{R,i}$ is outward unit normal vector on R with respect to $\pi(F_i)$.
(ii) $n_{R,i}$ is outward unit normal vector on R with respect to F_i.



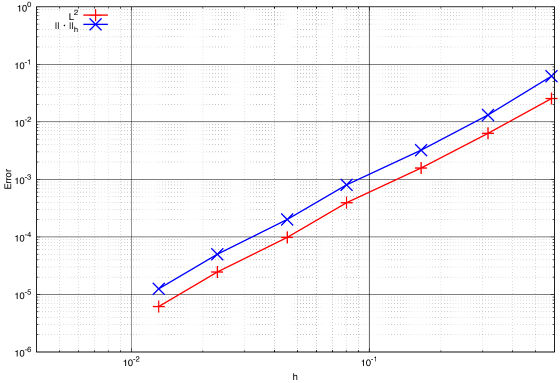
Similarly, for $R \in \mathcal{R}_h$,

$$\{\!\{v\}\!\}_R := \frac{1}{2}(v_1 + v_2), \qquad [\![v]\!]_R := v_1 n_{R,1} + v_2 n_{R,2},$$

$$\{\!\{\nabla_{\Gamma_h} v\}\!\}_R := \frac{1}{2}(\nabla_{\Gamma_h} v_1 + \nabla_{\Gamma_h} v_2), \qquad [\![\nabla_{\Gamma_h} v]\!]_R := \nabla_{\Gamma_h} v_1 \cdot n_{R,1} + \nabla_{\Gamma_h} v_2 \cdot n_{R,2}.$$

Here, there exist distinct F_1, $F_2 \in \mathcal{F}_h^{\partial}$ satsfying $R = F_1 \cap F_2$, $v_i = v|_{F_i}$.
(i) $n_{R,i}$ is outward unit normal vector on R with respect to $\pi(F_i)$.
(ii) $n_{R,i}$ is outward unit normal vector on R with respect to F_i.

$\Omega = \{|x| < 1\} \subset \mathbb{R}^2$, exact solution: $u(x) = \exp(-|x|^2)$

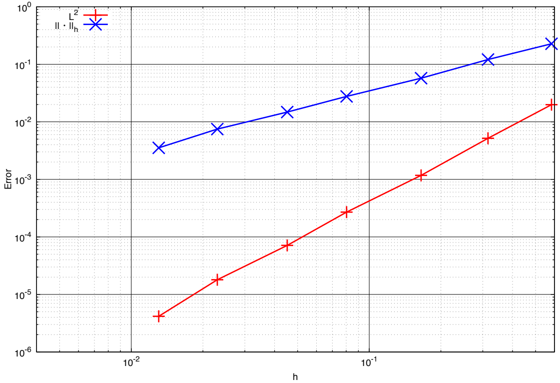


Energy and L^2 norm error using P2 element

Energy norm: $O(h^2)$, L^2 norm: $O(h^2)$

Appendix 2. Numerical examples (P2 element)

$\Omega = \{|x| < 1\} \subset \mathbb{R}^2$, exact solution: $u(x) = \sin(x_1)\sin(x_2)$



Energy and L^2 norm error using P2 element

Energy norm: $O(h^{1.5})$, L^2 norm: $O(h^2)$